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THESIS

PROBABILITY OF DETECTION FOR A GO CFAR RADAR PROCESSOR USING ENVELOPE DETECTION APPROXIMATION

by

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PROBABILITY OF DETECTION FOR A GO CFAR RADAR PROCESSOR USING ENVELOPE DETECTION APPROXIMATION

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Abstract

The greatest-of logic for a constant false alarm rate processor (GO CFAR) is a commonly used method for the adaptive setting of a radar detection threshold in the presence of clutter edges. Instead of using a true envelope detector $x = \sqrt{I^2 + Q^2}$, which is difficult to implement, envelope detection approximations of the form $x_e = aMax\{|I|, |Q|\} + bMin\{|I|, |Q|\}$ and $\hat{x}_e = a|I| + b|Q|$ (no *Max* and *Min* operators) where a and b are constants, are often used to detect a signal decomposed into its in-phase (I) and quadrature (Q) components. Closed form expressions are derived for the probability density function (pdf) of a radar range cell containing a detected target signal in the presence of noise using both approximations. These can then be used to calculate the probability of detection (P_D). They can also be used to calculate the probability of false alarm (P_{FA}) if the means of I and Q are set to zero. Closed form expressions are obtained analytically and by curve-fitting the numerically derived pdf of the target cell. Finally, using Monte Carlo techniques, this thesis also compares the GO CFAR detection performance using envelope detection approximations x_e, $\hat{x}_e$ and the true envelope detector x for zero mean white Gaussian noise input samples I and Q.

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Acknowledgment

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I. INTRODUCTION

A. CFAR SIGNAL PROCESSING

Signals processed by radars are inherently corrupted by environmental clutter and thermal noise from receiver components. The echo signal returned from any potential target will thus be accompanied by noise from these sources. The simplest method of detecting the target return signal is to set a fixed power amplitude threshold level which is known to be greater than the mean noise level and most of the noise peaks, but less than the expected magnitude of the target return signal. Thus, any signal which is measured to be greater than the threshold is declared by the radar to be a target detection, as shown in Figure 1. Such a detection may include a noise peak, and thus a false detection. This is known as a false alarm. By measuring the average number of false alarms occurring in a specified period of time, a false alarm rate can be established. The ratio of the average number of false alarms to the total number of noise samples is known as the probability of false alarm (P_{FA}). By the same token, the ratio of the average number of target signal detections to the total number of target signals sampled is known as the probability of detection (P_D). This implies that some target signals may not be detected, and thus the probability of a missed target detection can be computed. In general, the detection performance of a radar is stated in terms of the P_{FA} and the P_D.

Fixed threshold detection is inadequate for radar systems that have to detect the presence of targets within background environments that are more complex and less known than thermal noise alone. This type of detection produces large changes in false alarm rates, a very undesirable characteristic for radar systems intended for high target detection sensitivity and reliability. Also, the great number of detections that occur saturate and disrupt the tracking computer associated with the radar system. Thus detector design

criteria for modern military radars is to maintain a low and constant P_{FA} as well as a good P_D for a given signal-to-noise power ratio (SNR). Automatic detection systems which

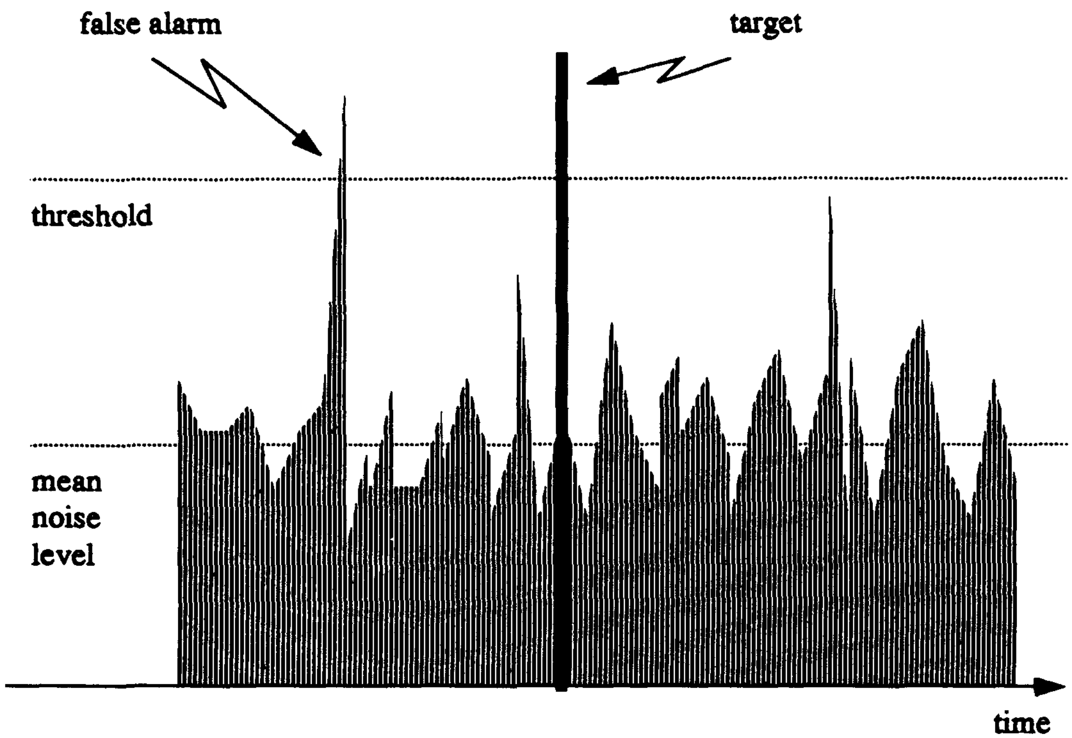

Figure 1: Detection Threshold

maintain a constant P_{FA} are known as constant false alarm rate (CFAR) systems. In these systems, the constant false alarm rate is maintained by adaptively varying the detection threshold level in accordance with estimates of the noise power (targets not present). Such detectors can easily cope with fluctuating noise levels, as the estimates are taken in real time. CFAR detection is usually performed in range. That is, the noise estimates are taken from a predetermined number of range bins (also known as cells) which are adjacent to the range bin on which detection is being performed (known as the test cell). This is shown in Figure 2. These adjacent cells are called reference cells. The analog-to-digital converter

samples taken over the time period occurring between two consecutive transmitted pulses are the range bins. That is, each range bin represents an increment in range from the radar.

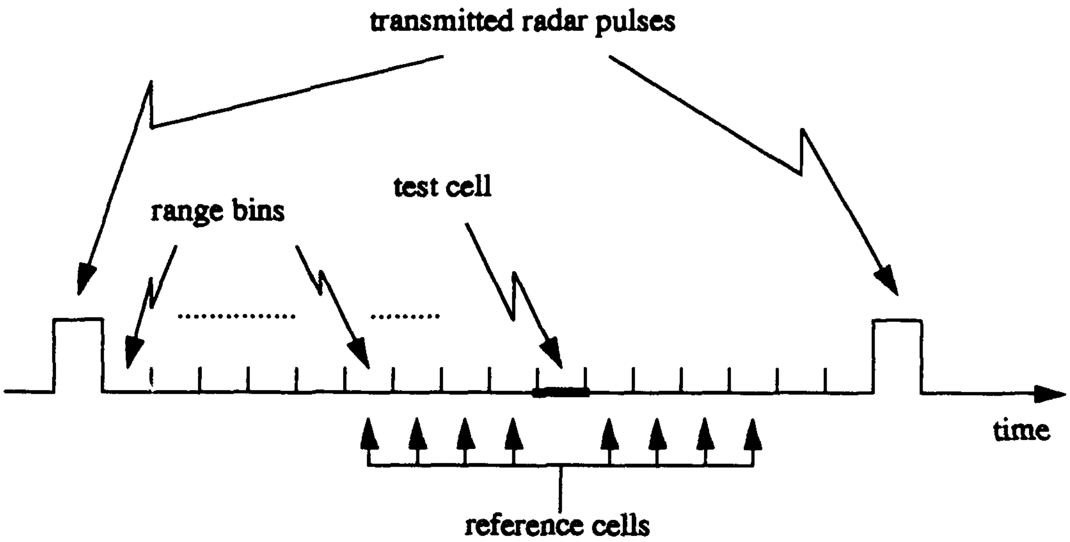

Figure 2: Unambiguous Radar Range with a Sample CFAR Window

The maximum possible range for the time period between the two pulses is known as the unambiguous range. The reference cells and test cell form a window which slides through a specified set of range bins, evaluating each range bin for a target.

Most adaptive threshold detectors assume that the samples in the reference range cells are independent and identically distributed. Furthermore, it is usually assumed that the time samples are independent. In essence, the detectors test whether the test cell power is sufficiently larger than the noise power in the reference cells. It is assumed that the probability density function of the noise is known except for a few unknown parameters. The surrounding reference cells are then used to estimate the unknown parameters, and a threshold based on the estimated parameters is obtained. Since the determined parameters are estimates, the resulting threshold has some error and must therefore be slightly larger

than the threshold that one would use if the parameters were known exactly *a priori*. This causes a loss in target detection sensitivity, called a CFAR loss, and thus results in a degradation of the P_D.

In order to analyze the P_D and the P_{FA} performances theoretically, the probability density functions of the radar range bins have to be derived. In general, given a density function for a non-target cell (noise only) and one for a target cell (signal with noise), the P_{FA} and the P_D are obtained by integrating over these functions for a given detection threshold T, as shown in Figure 3. Obtaining these functions can be very tedious. The

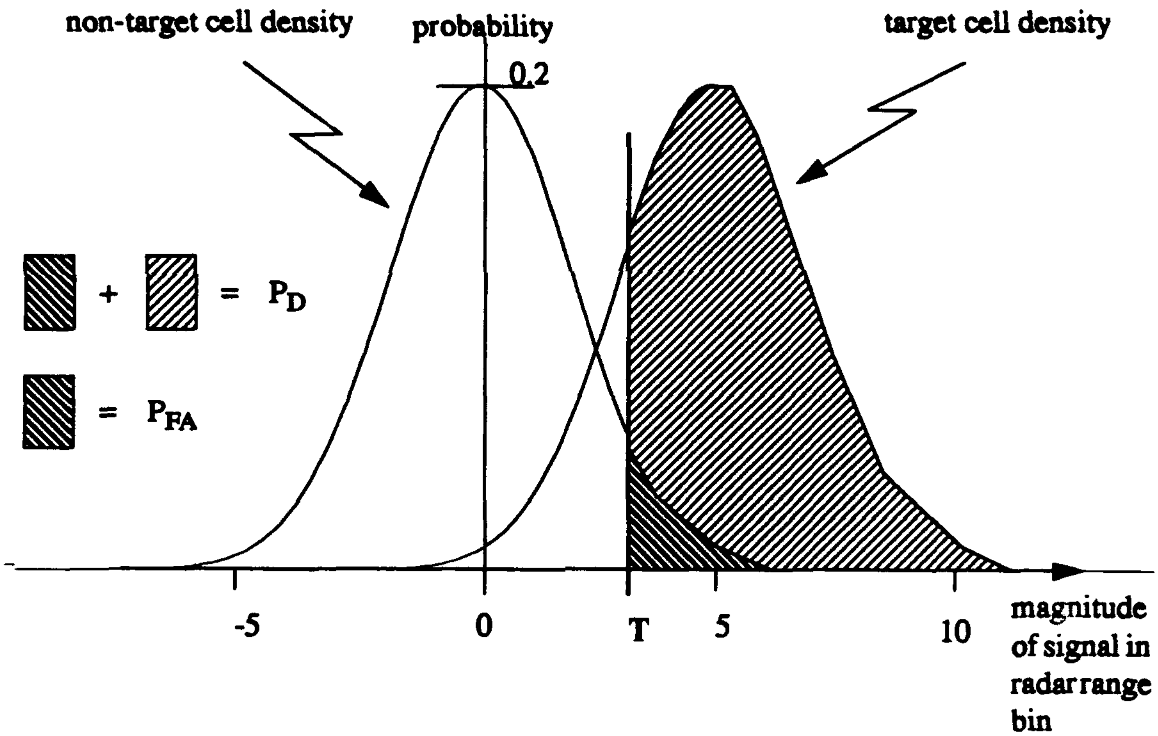

Figure 3: P_D and P_{FA}

resulting analytical expressions representing these density functions are not always practical to manipulate and program. Thus, attempts are often made to model the equations

with simple closed form expressions to facilitate the analysis and implementation of the desired CFAR processing scheme.

B. PRINCIPLE CONTRIBUTIONS

The main contribution of this research is the rigorous treatment of the effects of two envelope detection *approximations* on the statistical characteristics of the detected signals within a greatest-of CFAR (GO CFAR) signal processor. The first approximation is given by $x_e = aMAX\{|I|, |Q|\} + bMIN\{|I|, |Q|\}$, where I and Q are, respectively, the in-phase and quadrature samples of the input signal. The values a and b are simple multiplying constants. A closed form solution for the probability density function of this detector is shown to be untrackable analytically. An alternative method to obtain a closed form expression for the density function is developed. This approach curve-fits the results of a numerical convolution. These closed form expressions enable a quick implementation of the function within system simulation programs. Reference tables provide the necessary data for this method.

The second approximation investigated is given by $\hat{x}_e = a|I| + b|Q|$, a simplified form of x_e. An analytical closed form solution is presented, as well as a curve-fit solution similar to that obtained for x_e. This allows a comparison of the two envelope detection approximations in terms of their coefficients.

Finally, a comparison between the true envelope detector GO CFAR performance and the performance using x_e and $\hat{x}_e$ is made in terms of numerically derived probability density functions and the P_D. Results show that there is a significant difference in processor performance for the case of I and Q being zero mean white Gaussian noise with a non-fluctuating target.

C. THESIS OUTLINE

The objective of the research documented here is to investigate a method of computing the P_D analytically for a GO CFAR processor that uses the envelope detection

approximations $x_e = aMax\{ I, Q \} + bMin\{ I, Q \}$ and $\hat{x}_e = a\,I + b\,Q$. The main hurdle in doing this is to determine a closed form expression for the probability density function of a detected radar range cell with a target signal present in it.

Section II describes the operation of a GO CFAR in a simple manner. The functionality of it is described by stepping through all its operations sequentially, providing a brief discussion at each point. It then provides a short discussion of envelope detection approximation. Finally, the requirements for the analytical determination of for the GO CFAR detector is outlined. Section III provides the rigorous analytical development of the closed form solution for the probability density function of a target test cell using the envelope detection approximation x_e. It is a step-by-step walk-through of the derivation, presented first as a general result and then applied to the particular requirements of this research. The difficulties encountered and the assumptions made for every step of the derivation are described. This section also provides a closed form expression derived from curve-fitting the numerical convolution. Section IV presents the closed form solution for the probability density function of a target test cell using the envelope detection approximation $\hat{x}_e$. A closed form solution using curve-fitting techniques is also described. Section V presents a comparison of the two methods of envelope detection approximation discussed in Sections III and IV in terms of the P_D and the P_{FA}. Section VI provides concluding remarks on the success of this research. It also discusses the possibilities for continued research in this field, as well as different aspects of the problem which could be further studied.

D. REFERENCE OVERVIEW

The studies by P.E. Pace and L.L. Taylor [Ref. 1] concentrate on obtaining the P_{FA} analytically for an envelope detection approximation GO CFAR processor. The approximation used is $\hat{x}_e = a|I| + b|Q|$. They derive a closed form solution of the probability density function of a radar range cell with noise only in it. Being able to

integrate this function in closed form, they are able to produce P_{FA} curves numerically. The P_{FA} curves are derived as a function of a threshold multiplier and the number of reference cells used on each side of the test cell They also derive, through curve-fitting, closed form expressions for the P_{FA} and compare its performance to the numerical solution.

The technical memorandum by D.J. Wilson [Ref. 2] derives the P_D for a test cell with an envelope detection approximation $\hat{x}_e$ with $a = 1$ and $b = 1$. He provides a useful insight into the analysis of this type of approximation and the subsequent derivation of the P_D.

Another technical memorandum, this time by D.G. Loberger [Ref. 3], compares the performance of a cell-averaging CFAR processor (similar in concept to GO CFAR processing) for two different configurations. He analyzes the situation in which the test range cell is not used in the noise averaging of the reference cells. He also examines the situation in which the test cell is included in the averaging. The important result of this paper is to relate the threshold multipliers of each situation considered with a simple conversion formula. Using this result as a constraint guarantees the same P_{FA} and P_D performances for either of the envelope detection approximations.

A study by G.P. Laulusa [Ref. 4] evaluates the P_{FA} and P_D performances of a GO CFAR processor which includes the test cell along with the reference cells in the estimation of the noise level. He uses the envelope detection approximation $\hat{x}_e$ with unity multiplying coefficients and a normalized version of the probability density function of the test cell.

The paper by V.G. Hansen and J.H. Sawyers [Ref. 5] determines the CFAR loss of a GO CFAR scheme as compared to a simple cell-averaging CFAR. In the process of doing this they derive an integral expression for the P_D in terms of the probability density function of a greatest-of output block. This, in effect, defines the main building block for any P_D and P_{FA} computations on a GO CFAR processor, for any type of signal detector at the input of the processor and for any variation of the GO CFAR processing scheme.

II. GO CFAR OVERVIEW

A. IMPLEMENTATION

A typical GO CFAR processor is shown in Figure 4. It uses an envelope detector, a lead window of reference cells, a lag window of reference cells, a test cell, two adders, a maximum value detector, two multipliers and a binary comparator. The true envelope detector is $x = \sqrt{I^2 + Q^2}$, where I and Q represent the in-phase and quadrature samples, respectively, of the input signal. . The I and Q samples are obtained from an analog-to-digital converter (ADC), which produces the digital representations of the radar return signal. The ADC receives its analog I and Q signals from the synchronous detection and down conversion circuits.

The envelope detected signal samples (from the I and Q samples) are not actually written into a GO CFAR processor per say, but rather are stored in consecutive range cells (or bins). This is not depicted in Figure 4. The GO CFAR processor is a sliding window which reads in the signal samples of the necessary range cells for processing. The sliding window enables the GO CFAR processor to test each cell for potential targets. It is divided into lead and lag segments, each of length n cells. Selection of n depends on the desired quality of the noise estimate, and usually lies between 4 and 32.

The signal samples of the lead and lag windows are summed and the two results compared. The maximum sum is selected to calculate the estimate of the noise level. This value is then multiplied by a factor of $1/n$ to provide the final time-average noise estimate for the signal in the test cell. The estimate is then multiplied by T, a threshold multiplier, to establish a final detection threshold V_t. This multiplier is preset to a value which will yield a desired, constant P_{FA}. Finally, the detection threshold is compared to the signal present in the test cell. If the signal in the test cell is larger than the detection threshold, a target

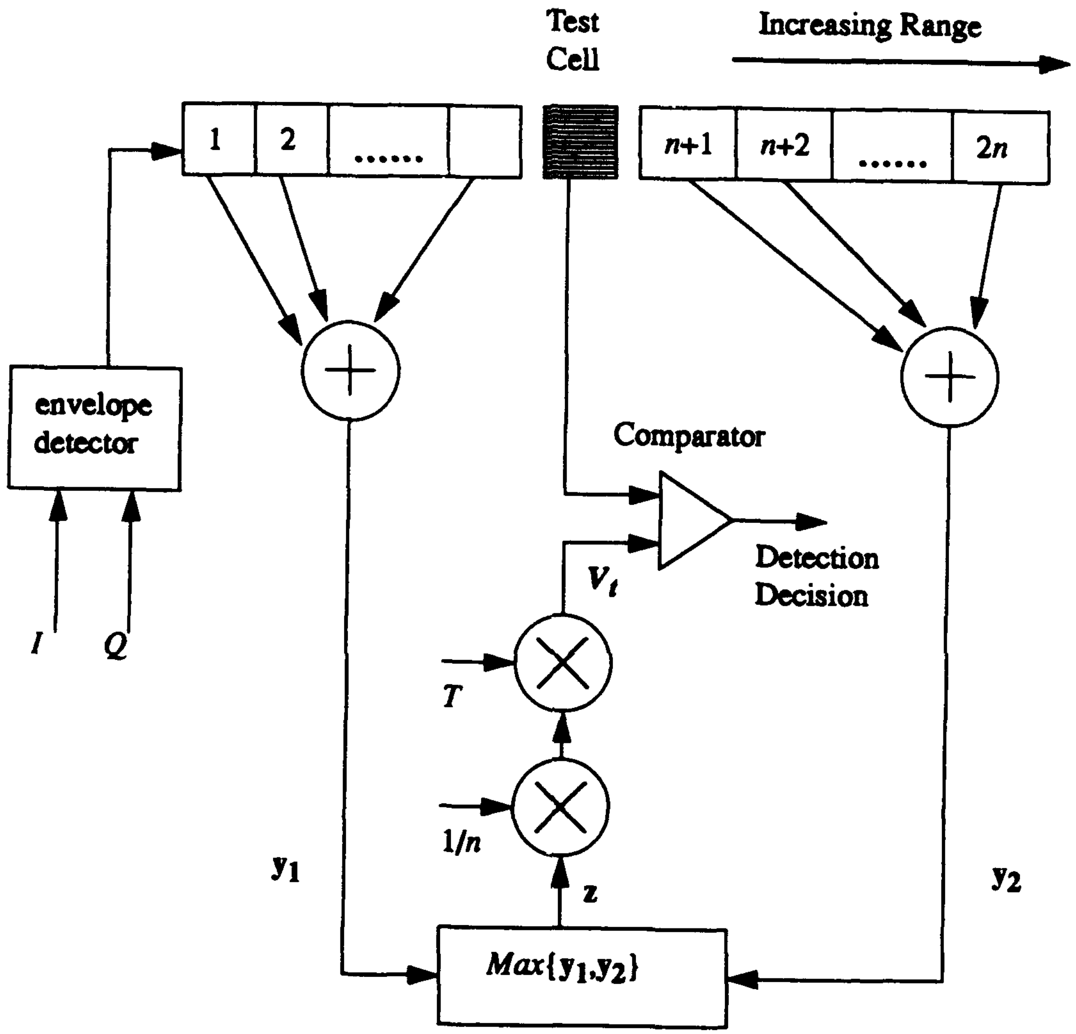

Figure 4: GO CFAR Processor

detection is declared. It can be concluded that GO CFAR processing is fairly simple in principle. However, it is much more complex to analyze theoretically.

B. ENVELOPE DETECTION APPROXIMATION

The envelope detector can be implemented with the envelope detection approximation mentioned in section I of the form

$$x_e = aMax\{|I|,|Q|\} + bMin\{|I|,|Q|\} \quad . \tag{1}$$

Another similar approximation is

$$\hat{x}_e = a|I| + b|Q| \, . \tag{2}$$

The constants a and b of the envelope detection approximation are predetermined so as to yield a desired performance level for the approximation given in (1). Envelope detection approximation is performed mainly for two reasons. First, much less hardware is required for implemention (less expensive). The squaring operation and the square-root operation in the true envelope detector are also difficult to design digitally so as to yield answers with minimal numerical rounding errors.

Secondly, envelope detection approximations perform faster in real time compared to the true envelope detector. This is a crucial requirement, since the sequential GO CFAR processing on all the range cells must be completed prior to the next received pulse. For this reason, GO CFAR processors are frequently found in low pulse repetition frequency (LPRF) radar modes of operation.

The *Max* and *Min* operators in the envelope detection approximation x_e have the

effect of uniformly distributing each complex input sample, of the form $S = I + jQ$, over the range of angles $\theta = 0 \ldots (\pi/4)$ radians. This dependence on θ can be expressed by rewriting the *Max* and *Min* terms in polar coordinates as

$$x = x_e (a \cdot \cos\theta + b \cdot \sin\theta) \quad . \tag{3}$$

The relative approximation error can be defined as

$$e(\theta) = \frac{x_e - x}{x_e} = 1 - (a \cdot \cos\theta + b \cdot \sin\theta) \quad . \tag{4}$$

Then, the performance criteria in terms of the average error and mean-square error can be expressed as, respectively,

$$\bar{e} = \frac{4}{\pi} \int_0^{\pi/4} e(\theta)\, d\theta \tag{5}$$

and

$$\bar{e}^2 = \frac{4}{\pi} \int_0^{\pi/4} e^2(\theta)\, d\theta \quad . \tag{6}$$

A number of envelope detection approximations are shown in Table 1. [Ref. 1].

The coefficients of Table 1 are considered in this research. This allows the findings from this thesis to be correlated to the results of previous research in this area [Ref. 1]. They provide a good variety of performances, in that the errors can be observed to vary over a wide range. The sixth approximation is such that it yields a zero average error.

TABLE 1: ENVELOPE DETECTION APPROXIMATIONS

approxi-mation	a	b	$\bar{e}$	$\bar{e}^2$
1	1	1	-27.3	30.0
2	1	1/2	-8.68	9.21
3	1	1/4	0.65	4.15
4	1	3/8	-4.02	4.76
5	31/32	3/8	-1.2	2.7
6	0.948	0.393	0	2.33
7	0.96043	0.39782	-1.3	2.7

C. ANALYTICAL DETERMINATION OF PROBABILITY OF DETECTION

It has been assumed for this paper that the I and Q input samples to the GO CFAR processor are independent and identically distributed. It is also assumed that the noise components of the samples are normally distributed zero mean and unity variance (~N(0,1)). Once the probability density function of the target test cell has been obtained, the probability of detection P_D can be determined using

$$P_D(SNR) = 2\int_0^{\infty} p_{y,n}(z) \left\{ \int_0^{z} p_{y,n}(y)\, dy \right\} \left\{ \int_{\frac{Tz}{n}}^{\infty} p_c(x, SNR)\, dx \right\} dz \quad . \tag{7}$$

The probability of false alarm P_{FA} is given by [Ref.1]

$$P_{FA} = 2\int_0^{\infty} p_{y,n}(z) \left\{ \int_0^{z} p_{y,n}(y)\, dy \right\} \left\{ \int_{\frac{Tz}{n}}^{\infty} p_c(x)\, dx \right\} dz \ . \tag{8}$$

The term $p_{y,n}$ represents the density function for the n reference cell summation and p_c represents the density function of the test cell. From Figure 4, $p_{y,n}$ is obtained by performing an n-fold convolution of the density function of a single reference cell. That is, each reference cell is assumed to have the same noise distribution and to be independent of each other. The distribution of the term p_c is the only difference between the P_D (SNR $\neq$ 0, target present) and the P_{FA} (SNR = 0, no target, noise only). In the former case it is dependent on SNR, whereas in the latter case it is not.

The true envelope detector would yield a Rician distribution (or non-central Rayleigh) for the target signal in each of the range cells. Closed from expressions for the

probability density function (pdf) of such a distribution exist and involve modified Bessel functions [Ref. 8]. The true envelope detector yields a Rayleigh distribution for noise alone. However, since the envelope detection approximations x_e and $\hat{x}_e$ are being used, the derivation of the pdf turns out to be much more difficult. This forms the focus of this research.

The goal is to attempt to determine if a closed form solution can be derived for the pdf of the signal produced by (1) and (2). The approach to be taken consists of deriving the probability density functions of the smallest components of the equation first. Then the transformed density functions have to be derived following the same constructions that make up (1) and (2). The pdf of x_e is $p_c(x)$, representing the pdf of a range cell with a target signal present in it. The pdf of $\hat{x}_e$ is $p_c(\hat{x})$. It is assumed that the target signal is located within the test cell only, and does not contribute power to the noise-only cells adjacent to it.

III. PDF FOR THE ENVELOPE DETECTION APPROXIMATION $aMax\{|I|,|Q|\}$ + $bMin\{|I|,|Q|\}$ OF A TARGET TEST CELL

The development outlined in this section derives the density function $p_c(x)$ of a target test cell sequentially. It begins with the probability density functions of each of the I and Q sample expressions. These are known to be Gaussian distributed, being composed of known means and normal noise. The densities of the magnitudes of I and Q are found next. Then the density functions of the terms $Max\{|I|,|Q|\}$ and $Min\{|I|,|Q|\}$ are derived. This is followed by the derivation of the density functions of these terms multiplied by constants, i.e. $aMax\{|I|,|Q|\}$ and $bMin\{|I|,|Q|\}$. Finally, the probability density function of the term $x_e = aMax\{|I|,|Q|\} + bMin\{|I|,|Q|\}$ is derived. The derivation of $p_c(x)$ involves a convolution of the two terms in x_e. Because the density functions of *Max* and *Min* operations are quite involved, the convolution of their terms generate a very complex probability density function. This may not be the most desirable approach. As shown, this density function is such that computational efficiency and computer memory capacity become important concerns. Additionally, in order to obtain a closed form solution of the convolution integral, curve-fitting techniques are used to model the error function. Instead of using what would turn out to be an extremely complex closed form solution for $p_c(x)$, curve-fitting techniques are also investigated to generate expressions for $p_c(x)$. The expressions for $p_c(x)$ are derived for the selections of a and b given in Table 1. The coefficients necessary to implement these expressions are tabulated to provide a useful design tool for the evaluation of the P_D and to provide a performance comparison with other envelope approximations such as $\hat{x}_e$.

A. CLOSED FORM PDF BY ANALYTICAL METHODS

The inputs into the envelope detection approximation block of Figure 4 are the in-phase and quadrature components of the radar return signal. The data sample for each channel can be expressed as the sum of a modulated amplitude and some noise, as

$$I = A\cos\theta + n_I \tag{9}$$

and

$$Q = A\sin\theta + n_Q \tag{10}$$

where A is the time-invariant signal amplitude and θ is the signal phase. The noise variables, n_I and n_Q, are assumed to be mutually independent and identically distributed Gaussian random variables with zero mean and unity variance (~N(0,1)). By definition, the I and Q terms are therefore also normally distributed random variables which are independent of each other and identically distributed. As discussed in section II, the signal phase is uniformly distributed over the range $0\ldots(\pi/4)$. The final result for the probability of detection is averaged over the phase. That is, the P_D is calculated as

$$P_D(SNR) = \frac{8}{\pi}\int_0^{\pi/4}\left[\int_0^{\infty} p_{y,n}(z)\left\{\int_0^{z} p_{y,n}(y)\,dy\right\}\left\{\int_{\frac{Tz}{n}}^{\infty} p_c(x, SNR, \theta)\,dx\right\}dz\right]d\theta\ . \tag{11}$$

The average values of I and Q are, respectively,

$$m_I = A\cos\theta \tag{12}$$

and

$$m_Q = A\sin\theta \quad \cdot \tag{13}$$

That is, the noise samples have zero mean. The signal amplitude is also related to the signal-to-noise ratio (SNR) as $A = \sqrt{2\sigma^2 SNR}$.

1. PDF of a Gaussian Random Variable

The probability density functions of I and Q are Gaussian and are expressed as

$$p_I(i) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left[\frac{-(i-m_I)^2}{2\sigma^2}\right]} \tag{14}$$

and

$$p_Q(q) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left[\frac{-(q-m_Q)^2}{2\sigma^2}\right]} \quad . \tag{15}$$

Since the variance is unity, the σ^2 terms can be omitted in the following derivations without any effect.

2. PDF of the Absolute Value of a Gaussian Random Variable

Considering only the I random variable, the cumulative distribution function (cdf) of I is the integration of the probability density function and can be expressed as

$$P_I(i) = Pr\{I \le i\}$$

or

$$P_I(i) = \int_{-\infty}^{i} e^{\left[\frac{-(u-m_I)^2}{2}\right]} du \tag{16}$$

which is the probability that a given realization of the random variable I will not exceed a threshold i. The cdf of $|I|$ is then

$$P_{|I|}(i) = Pr\{\{(-I) \geq (-i)\} \cap \{I \leq i\}\}$$

$$= Pr\{|I| \leq i\}$$

or

$$P_{|I|}(i) = \frac{1}{\sqrt{2\pi}} \int_{-i}^{i} e^{\left[\frac{-(u-m_I)^2}{2}\right]} du \ . \tag{17}$$

It is known that the probability density function of $|I|$ is obtained by differentiating the cumulative distribution function of $|I|$. Since the integral operation is linear, (17) can be expanded into two parts as

$$P_{|I|}(i) = \int_{-i}^{i} p_I(u)\, du$$

or

$$P_{|I|}(i) = \int_{-i}^{0} p_I(u)\, du + \int_{0}^{i} p_I(u)\, du \ . \tag{18}$$

Differentiating this expansion with respect to i, the density function of $|I|$ can be expressed as

$$p_{|I|}(i) = \frac{d}{di}\int_{-i}^{0} p_I(u)\,du + \frac{d}{di}\int_{0}^{i} p_I(u)\,du \tag{19}$$

and becomes

$$p_{|I|}(i) = p_I(i) + p_I(-i) \ . \tag{20}$$

The final expression for the pdf of $|I|$ is then given by

$$p_{|I|}(i) = \frac{1}{\sqrt{2\pi}}\left\{e^{\left[\frac{-(i-m_I)^2}{2}\right]} + e^{\left[\frac{-(-i-m_I)^2}{2}\right]}\right\}u(i)$$

or

$$p_{|I|}(i) = \frac{1}{\sqrt{2\pi}}\left\{e^{\left[\frac{-(i-m_I)^2}{2}\right]} + e^{\left[\frac{-(i+m_I)^2}{2}\right]}\right\}u(i) \tag{21}$$

where $u(i)$ is the unit step function. This has to be present to ensure that the variable i, representing values of $|I|$, remains positive in accordance with the absolute value operation. In a similar fashion , the probability density function of $|Q|$ can be expressed as

$$p_{|Q|}(q) = \frac{1}{\sqrt{2\pi}}\left\{e^{\left[\frac{-(q-m_Q)^2}{2}\right]} + e^{\left[\frac{-(q+m_Q)^2}{2}\right]}\right\}u(q) \ . \tag{22}$$

These functions are valid probability density functions since they satisfy the appropriate requirements: they are positive for any value of i and q, and they both integrate to unity.

3. PDF of $Max\{|I|,|Q|\}$

To begin, a random variable Z is defined which takes the maximum of two other mutually independent random variables X and Y,

$$Z = MAX(X, Y) \tag{23}$$

The variable Z has a cdf defined in the following manner

$$F_Z(z) = Pr\{Z \le z\}$$

$$= Pr\{MAX(X, Y) \le z\}$$

or

$$F_Z(z) = Pr\{X \le z \cap Y \le z\} \tag{24}$$

In other words, the cumulative distribution function of Z is the probability that both $X \le z$ *and* $Y \le z$ for any given value z. Because X and Y are independent from each other,

$$F_Z(z) = Pr\{X \le z\} Pr\{Y \le z\}$$

or

$$F_Z(z) = F_X(z) F_Y(z) \tag{25}$$

That is, the joint cdf of the two independent random variables X and Y is equal to the product of their respective distribution functions, $F_X(z) \bullet F_Y(z)$.

In general, the derivative of a function which is composed of the product of two other functions (u and v, for example) which in turn each depend on the same variable (x) can be written as [Ref. 9]

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx} \quad . \tag{26}$$

Applying (26) to (25) to obtain the density function of Z, and knowing that differentiating a distribution function with respect to its argument yields the corresponding density function, $f_Z(z)$ is obtained as

$$\frac{d}{dz}F_Z(z) = f_Z(z) = \frac{d}{dz}[F_X(z)F_Y(z)]$$

$$= F_X(z)\frac{d}{dz}F_Y(z) + F_Y(z)\frac{d}{dz}F_X(z)$$

$$= F_X(z)f_Y(z) + F_Y(z)f_X(z)$$

or

$$f_Z(z) = f_X(z)F_Y(z) + f_Y(z)F_X(z) \quad . \tag{27}$$

This expression represents the pdf of a maximum of two random variables. It can also be

derived using [Ref. 8]

$$f_Z(z) = \int\int_{\Delta D_z} f(x, y)\, dxdy \tag{28}$$

where the term ΔD_Z is defined as the region of the x-y plane where $z < Z \leq z + dz$, and the function $f(x,y)$ is the joint density function defined as

$$f(x, y) = f_X(x) f_Y(y) \tag{29}$$

because of the fact that the two random variables are independent. This approach, however, is slightly more complicated.

In the context of this thesis research, the random variable Z is defined as

$$Z = Max(|I|, |Q|) \tag{30}$$

In order to find its pdf using (27), the cumulative distribution functions of the $|I|$ and $|Q|$ random variables are first derived. By definition, these are obtained by integrating their respective density functions over the range $-\infty$ to z. Using (21), the cdf of $p_{|I|}(i)$ is given by

$$P_{|I|}(z) = \int_{-\infty}^{z} p_{|I|}(i)\, di$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \left\{ e^{\left[\frac{-(i-m_I)^2}{2}\right]} + e^{\left[\frac{-(i+m_I)^2}{2}\right]} \right\} u(i)\, di$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^z \left\{ e^{\left[\frac{-(i-m_I)^2}{2}\right]} + e^{\left[\frac{-(i+m_I)^2}{2}\right]} \right\} di$$

or

$$P_{|I|}(z) = \frac{1}{2}\left[erf\left(\frac{z-m_I}{\sqrt{2}}\right) + erf\left(\frac{z+m_I}{\sqrt{2}}\right) \right] \quad . \tag{31}$$

The function *erf* is the error function defined as

$$erf\left(\frac{x-m}{\sigma}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{x-m}{\sigma}} e^{\frac{-t^2}{2}} dt \quad . \tag{32}$$

Similarly for $|Q|$, using (22), the cumulative distribution function for $p_{|Q|}$ is given by

$$P_{|Q|}(z) = \frac{1}{2}\left[erf\left(\frac{z-m_Q}{\sqrt{2}}\right) + erf\left(\frac{z+m_Q}{\sqrt{2}}\right) \right] \quad . \tag{33}$$

Using (27), the density function of Z is then

$$p_Z(z) = p_{|I|}(z) P_{|Q|}(z) + p_{|Q|}(z) P_{|I|}(z)$$

or

$$p_Z(z) = \frac{1}{2\sqrt{2\pi}} \left[\left\{ e^{\left[\frac{-(z-m_I)^2}{2}\right]} + e^{\left[\frac{-(z+m_I)^2}{2}\right]} \right\} \{ erf(\frac{z-m_Q}{2}) + erf(\frac{z+m_Q}{2}) \} + \right.$$

$$\left\{ e^{\left[\frac{-(z-m_Q)^2}{2}\right]} + e^{\left[\frac{-(z+m_Q)^2}{2}\right]} \right\} \{ erf(\frac{z-m_I}{2}) + erf(\frac{z+m_I}{2}) \} \Bigg] u(z) \quad . \tag{34}$$

That is, the probability density function of the maximum of the two random variables $|I|$ and $|Q|$ is a combination of exponentials and error functions.

4. PDF of $Min\{|I|,|Q|\}$

The density function of a minimum of two mutually independent random variables is found in a similar manner as that for a maximum, discussed in the previous subsection. The random variable W is assigned to be

$$W = Min(X, Y) \tag{35}$$

where X and Y are two independent random variables. The cumulative distribution function of W is expressed as

$$F_W(w) = Pr\{W \le w\}$$

$$= Pr\{MIN(X, Y) \le w\}$$

or

$$F_W(w) = Pr\{X \le w \cup Y \le w\} \quad . \tag{36}$$

This means that the cdf of the minimum of X and Y is equal to the probability that either $X \leq w$ *or* $Y \leq w$ for any given value w of W. Thus the distribution function can be expanded as per the definition of an *or* operator as follows

$$F_W(w) = Pr\{X \leq w\} + Pr\{Y \leq w\} - Pr\{X \leq w \cap Y \leq w\} \quad . \tag{37}$$

Because the random variables X and Y are independent, the cumulative distribution function becomes

$$F_W(w) = Pr\{X \leq w\} + Pr\{Y \leq w\} - Pr\{X \leq w\} Pr\{Y \leq w\}$$

or

$$F_W(w) = F_X(w) + F_Y(w) - F_X(w) F_Y(w) \tag{38}$$

which is expressed in terms of the distribution functions of each of the random variables X and Y. Differentiating the distribution function of the random variable W with respect to w, and thereby applying equation (26) to the third term, the probability density function of W can be written as follows

$$\frac{d}{dw} F_W(w) = f_W(w)$$

$$= \frac{d}{dw}[F_X(w) + F_Y(w) - F_X(w) F_Y(w)]$$

$$= \frac{d}{dw} F_X(w) + \frac{d}{dw} F_Y(w) - \frac{d}{dw}[F_X(w) F_Y(w)]$$

$$= \frac{d}{dw}F_X(w) + \frac{d}{dw}F_Y(w) - \left[F_X(w)\frac{d}{dw}F_Y(w) + F_Y(w)\frac{d}{dw}F_X(w)\right]$$

$$= f_X(w) + f_Y(w) - [F_X(w)f_Y(w) + F_Y(w)f_X(w)]$$

$$= [f_X(w) - f_X(w)F_Y(w)] + [f_Y(w) - f_Y(w)F_X(w)]$$

or

$$f_W(w) = f_X(w)\,[1 - F_Y(w)] + f_Y(w)\,[1 - F_X(w)] \quad . \qquad (39)$$

In this thesis it is desired to find the density function of the random variable

$$W = Min(|I|, |Q|) \qquad (40)$$

In accordance with (39), the density function of W is given by

$$p_W(w) = p_{|I|}(w)\,[1 - P_{|Q|}(w)] + p_{|Q|}(w)\,[1 - P_{|I|}(w)] \quad . \qquad (41)$$

Using (31) and (33) for the cumulative distribution functions of the random variables $|I|$ and $|Q|$ respectively, and (21) and (22) for the density functions of $|I|$ and $|Q|$ respectively, the probability density function of W can then be expressed as

$$p_W(w) =$$

$$\frac{1}{\sqrt{2\pi}}\left[\left\{e^{\left[\frac{-(w-m_I)^2}{2}\right]}+e^{\left[\frac{-(w+m_I)^2}{2}\right]}\right\}\{1-\frac{1}{2}erf\left(\frac{w-m_Q}{\sqrt{2}}\right)-\frac{1}{2}erf\left(\frac{w+m_Q}{\sqrt{2}}\right)\}+\right.$$

$$\left.\left\{e^{\left[\frac{-(w-m_Q)^2}{2}\right]}+e^{\left[\frac{-(w+m_Q)^2}{2}\right]}\right\}\{1-\frac{1}{2}erf\left(\frac{w-m_I}{\sqrt{2}}\right)-\frac{1}{2}erf\left(\frac{w+m_I}{\sqrt{2}}\right)\}\right]u(w)$$

or

$$p_W(w) =$$

$$=\frac{1}{2\sqrt{2\pi}}\left[\left\{e^{\left[\frac{-(w-m_I)^2}{2}\right]}+e^{\left[\frac{-(w+m_I)^2}{2}\right]}\right\}\{2-erf\left(\frac{w-m_Q}{\sqrt{2}}\right)-erf\left(\frac{w+m_Q}{\sqrt{2}}\right)\}+\right.$$

$$\left.\left\{e^{\left[\frac{-(w-m_Q)^2}{2}\right]}+e^{\left[\frac{-(w+m_Q)^2}{2}\right]}\right\}\{2-erf\left(\frac{w-m_I}{\sqrt{2}}\right)-erf\left(\frac{w+m_I}{\sqrt{2}}\right)\}\right]u(w) \quad . \tag{42}$$

This defines the probability density function of the minimum of the random variables $|I|$ and $|Q|$, expressed in terms of exponentials and error functions. Note that the unit step function prevents $p_W(w)$ from being negative.

5. PDF of $aMax\{|I|,|Q|\}$ and $bMin\{|I|,|Q|\}$

Generally, if the random variable Y is a linear transformation of one other random variable X of the form

$$Y = aX \tag{43}$$

where a is a constant, the probability density function of Y is then given by [Ref. 8]

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y}{a}\right) \tag{44}$$

where f_X is the density function of X. Let the random variable Z' be given by

$$Z' = aZ = aMax\{|I|, |Q|\} \tag{45}$$

The density function $p_Z(z)$ of Z is given by (34). Assume the constant a is always positive. Therefore, using (44), the density function of Z' is then

$$p_{Z'}(z) = \frac{1}{a} p_Z\left(\frac{z}{a}\right)$$

or

$$p_{Z'}(z) = \frac{1}{a2\sqrt{2\pi}}\left[\left\{e^{\left[\frac{-(\frac{z}{a}-m_I)^2}{2}\right]} + e^{\left[\frac{-(\frac{z}{a}+m_I)^2}{2}\right]}\right\}\left\{erf\left(\frac{\frac{z}{a}-m_Q}{2}\right) + erf\left(\frac{\frac{z}{a}+m_Q}{2}\right)\right\} + \right.$$

$$\left\{ e^{\left[\frac{-(\frac{z}{a}-m_Q)^2}{2}\right]} + e^{\left[\frac{-(\frac{z}{a}+m_Q)^2}{2}\right]} \right\} \left\{ erf\left(\frac{\frac{z}{a}-m_I}{2}\right) + erf\left(\frac{\frac{z}{a}+m_I}{2}\right) \right\} \Bigg] u(z) \quad .$$

(46)

In a similar fashion, let W' be given by

$$W' = bW = bMin\{|I|, |Q|\} \tag{47}$$

where b is a constant. The density function $p_W(w)$ of W is given by equation (42). Assuming that the constant b is always positive, and again using (44), the probability density function of W' is given by

$$p_{W'}(w) = \frac{1}{b} p_W(\frac{w}{b})$$

or

$$p_{W'}(w) =$$

$$\frac{1}{b2\sqrt{2\pi}} \Bigg[\left\{ e^{\left[\frac{-(\frac{w}{b}-m_I)^2}{2}\right]} + e^{\left[\frac{-(\frac{w}{b}+m_I)^2}{2}\right]} \right\} \left\{ 2 - erf\left(\frac{\frac{w}{b}-m_Q}{\sqrt{2}}\right) - erf\left(\frac{\frac{w}{b}+m_Q}{\sqrt{2}}\right) \right\} +$$

$$\left\{ e^{\left[\frac{-(\frac{w}{b}-m_Q)^2}{2}\right]} + e^{\left[\frac{-(\frac{w}{b}+m_Q)^2}{2}\right]} \right\} \left\{ 2 - erf\left(\frac{\frac{w}{b}-m_I}{\sqrt{2}}\right) - erf\left(\frac{\frac{w}{b}+m_I}{\sqrt{2}}\right) \right\} \Bigg] u(w) \quad .$$

(48)

6. PDF of $aMax\{|I|,|Q|\} + bMin\{|I|,|Q|\}$

The envelope detection approximation x_e is also a random variable and is given by

$$x_e = aMAX\{|I|, |Q|\} + bMIN\{|I|, |Q|\}$$

or

$$x_e = Z' + W' \quad . \tag{49}$$

The density functions of the random variables Z' and W' are known and given by (46) and (48), respectively. Since I and Q are assumed to be independent random variables, then Z' and W' are also independent. This is known because of the fact that at any given time, if Z' is a function of $|I|$, then W' is automatically a function of $|Q|$ only, and conversely if Z' is a function of $|Q|$, then W' is a function of $|I|$ only, by definition of the *Max* and *Min* operations in x_e.

If two random variables are independent, then the density of their sum equals the convolution of their densities [Ref. 8]. Hence the density function of x_e is the convolution of the density functions Z' and W'. This represents the density function $p_c(x)$ of a target test cell. The pdf of x_e is thus given by

$$p_c(x) = p_{Z'}(x) \otimes p_{W'}(x)$$

or

$$p_c(x) = \int_{-\infty}^{\infty} p_{Z'}(y) p_{W'}(x-y)\, dy \qquad (50)$$

This linear convolution integral must be solved to obtain the pdf of x_e. Since $p_{Z'}(z) = 0$ for $z < 0$ and $p_{W'}(w) = 0$ for $w < 0$, then $p_c(x) = 0$ for $x < 0$. In addition to removing the need for the unit step function in the density function expressions of Z' and W', the integration limits can be set from 0 to x. The density function of x_e then becomes

$$p_c(x) = \int_0^x \left\{ \frac{1}{a2\sqrt{2\pi}} \left[\left\{ e^{\left[\frac{-(\frac{y}{a}-m_I)^2}{2}\right]} + e^{\left[\frac{-(\frac{y}{a}+m_I)^2}{2}\right]} \right\} \times \right.\right.$$

$$\left\{ erf\left(\frac{\frac{y}{a}-m_Q}{2}\right) + erf\left(\frac{\frac{y}{a}+m_Q}{2}\right) \right\} +$$

$$\left.\left. \left\{ e^{\left[\frac{-(\frac{y}{a}-m_Q)^2}{2}\right]} + e^{\left[\frac{-(\frac{y}{a}+m_Q)^2}{2}\right]} \right\} \left\{ erf\left(\frac{\frac{y}{a}-m_I}{2}\right) + erf\left(\frac{\frac{y}{a}+m_I}{2}\right) \right\} \right] \right\} \times$$

$$\left\{ \frac{1}{b2\sqrt{2\pi}} \left[\left\{ e^{\left[\frac{-(\frac{x-y}{b}-m_I)^2}{2}\right]} + e^{\left[\frac{-(\frac{x-y}{b}+m_I)^2}{2}\right]} \right\} \times \right.\right.$$

$$\left\{2-erf\left(\frac{\frac{x-y}{b}-m_Q}{\sqrt{2}}\right)-erf\left(\frac{\frac{x-y}{b}+m_Q}{\sqrt{2}}\right)\right\}+$$

$$\left\{e^{\left[\frac{-(\frac{z-y}{b}-m_Q)^2}{2}\right]}+e^{\left[\frac{-(\frac{z-y}{b}+m_Q)^2}{2}\right]}\right\}\times$$

$$\left\{2-erf\left(\frac{\frac{z-y}{b}-m_I}{\sqrt{2}}\right)-erf\left(\frac{\frac{z-y}{b}+m_I}{\sqrt{2}}\right)\right\}\left.\right]\left.\right\}dy \quad . \tag{51}$$

As demonstrated, this probability density function is a complicated expression. An evaluation of this expression is presented next. Each term within the integral is expanded. This expansion is necessary to enable a closed form solution to be developed. The pdf of x_e therefore becomes

$$p_c(z) = \frac{1}{8ab\pi}\int_0^z [\,(-2EQ-2EH+2BJ+2BS-2EO+2LO-2KA+2BW+$$

$$KIR+2LM-2KW+2LN-2KU-2EG-LDG-BXW-BDW+2LG-$$

$$LPN+KPS+2LQ-2KV-BDA-2EM+EIO+KIV-LIH+KPU-$$

$$BPS+EXM-LIQ-LIO+KDF+KXF-LPQ+KXA+EPH+EXC+$$

$$2LC-2KF-2ET-LDM-LXC+KIU+KXJ-LXT-LPO-LXM-$$

$$BIR - BPR - LIN - BIS + EPN - BXA - LPH + KPV + KDA - LDT -$$

$$LXG - BXJ + KIS - BDJ + EXG + EXT - 2KJ + 2LT + KDW - BXF -$$

$$BPV - BIU - BPU - BIV + KXW + 2BV + 2BU + 2BR - 2EN + 2LH +$$

$$KPR + EDT - BDF - 2KS - 2EC - 2KR + EIQ + EPQ + EPO + EIN +$$

$$EDM - LDC + KDJ + EIH + EDG + 2BF + EDC + 2BA\ ) \times$$

$$e^{\frac{-1}{2a^2b^2}(-2xua^2 + u^2b^2 + 3m_Q^2a^2b^2 + x^2a^2 + u^2a^2 + 2um_Qa^2b + 2zm_Q^2a^2b +}$$

$$2xm_Ia^2b + 3m_I^2a^2b^2 + 2um_Ia^2b + 2um_Iab^2 + 2um_Qab^2\ )\ \Big]\, du \qquad (52)$$

The variables $A \ldots X$ in equation (52) are symbols which are represented as

$$A = e^{\frac{1}{ab}(2um_Qb + 2um_Ia + m_Q^2ab + um_Qa + xm_Qa + m_I^2ab + um_Ib)} \qquad (53)$$

$$B = erf\left(\frac{u + m_Ia}{a\sqrt{2}}\right) \qquad (54)$$

$$C = e^{\frac{1}{2ab}(4um_Ib + 4xm_Ia + 3m_Q^2ab + 2um_Qa + 2xm_Qa + m_I^2ab + 2um_Qb)} \qquad (55)$$

$$D = erf\left(\frac{x - u + m_Qb}{b\sqrt{2}}\right) \qquad (56)$$

$$E = erf\left(\frac{-u + m_Qa}{a\sqrt{2}}\right) \qquad (57)$$

$$F = e^{\frac{1}{ab}(2um_Qb + 2xm_Ia + m_Q^2ab + um_Qa + xm_Qa + m_I^2ab + um_Ib)} \tag{58}$$

$$G = e^{\frac{1}{2ab}(4xm_Ia + 3m_Q^2ab + 2um_Qa + 2xm_Qa + m_I^2ab + 2um_Qb)} \tag{59}$$

$$H = e^{\frac{1}{ab}(2um_Ib + 2xm_Qa + m_Q^2ab + m_Iax + um_Ia + m_I^2ab + um_Qb)} \tag{60}$$

$$I = erf\left(\frac{x - u + m_Ib}{b\sqrt{2}}\right) \tag{61}$$

$$J = e^{\frac{1}{ab}(2um_Ia + m_Q^2ab + um_Qa + m_Qax + m_I^2ab + um_Ib)} \tag{62}$$

$$K = erf\left(\frac{-u + m_Ia}{a\sqrt{2}}\right) \tag{63}$$

$$L = erf\left(\frac{u + m_Qa}{a\sqrt{2}}\right) \tag{64}$$

$$M = e^{\frac{1}{2ab}(4um_Ib + 4um_Ia + 3m_Q^2ab + 2um_Qa + 2xm_Qa + m_I^2ab + 2um_Qb)} \tag{65}$$

$$N = e^{\frac{1}{ab}(2um_Ib + 2um_Qa + m_Q^2ab + m_Iax + um_Ia + m_I^2ab + um_Qb)} \tag{66}$$

$$O = e^{\frac{1}{ab}(2um_Qa + m_Q^2ab + m_Iax + um_Ia + m_I^2ab + um_Qb)} \tag{67}$$

$$P = erf\left(\frac{z - u - m_Ib}{b\sqrt{2}}\right) \tag{68}$$

$$Q = e^{\frac{1}{ab}(2xm_Qa + m_Q^2ab + m_Iax + um_Ia + m_I^2ab + um_Qb)} \tag{69}$$

$$R = e^{\frac{1}{2ab}(4xm_Qa + m_Q^2ab + 2xm_Ia + 3m_I^2ab + 2um_Ia + 2um_Ib)} \tag{70}$$

$$S = e^{\frac{1}{2ab}(4um_Qa + m_Q^2ab + 2xm_Ia + 3m_I^2ab + 2um_Ia + 2um_Ib)} \tag{71}$$

$$T = e^{\frac{1}{2ab}(4um_Ia + 3m_Q^2ab + 2um_Qa + 2xm_Qa + m_I^2ab + 2um_Qb)} \tag{72}$$

$$U = e^{\frac{1}{2ab}(4um_Qb + 4xm_Qa + m_Q^2ab + 2xm_Ia + 3m_I^2ab + 2um_Ia + 2um_Ib)} \tag{73}$$

$$V = e^{\frac{1}{2ab}(4um_Qb + 4um_Qa + m_Q^2ab + 2xm_Ia + 3m_I^2ab + 2um_Ia + 2um_Ib)} \tag{74}$$

$$W = e^{\frac{1}{ab}(2xm_Ia + m_Q^2ab + um_Qa + m_Qax + m_I^2ab + um_Ib)} \tag{75}$$

$$X = erf\left(\frac{x - u - m_Qb}{b\sqrt{2}}\right) \quad . \tag{76}$$

a. *Error Function Approximation*

The density function of the envelope detection approximation x_e given by (52) cannot be obtained in a mathematical closed form. The specific difficulty in doing this is to integrate expressions of the form

$$\int \{erf(au + b)\, e^{cu^2 + du + e}\}\, du \tag{77}$$

or

$$\int \{erf(au + b)\, erf(cu + d)\, e^{eu^2 + fu + g}\}\, du \quad . \tag{78}$$

A few closed form solutions exist for indefinite integrals of an error function multiplying an exponential, but their variable arguments are simpler and different in form from those of (77). Very accurate series expansions offering error residuals to the order of 10^{-5} to

10^{-10} also exist for the error function. These expressions, however, are of fifth or higher order, and are thus very difficult, if not impossible, to integrate in a closed form. [Ref. 9]

The simplest approach available to integrate (52) in closed form is to approximate the error function with a polynomial. A second order polynomial-fitted approximation will ensure a closed form solution for the integration of terms in the form of (77) and (78). The error function can thus be approximated with a maximum residual error magnitude of less than 1.75e-3. In order to achieve this order of residual, the error function *erf(x)* has to be modelled in six segments. For the first five segments the error function is approximated by polynomials of the form

$$erf(x) \approx ax^2 + bx + c \tag{79}$$

where the coefficients a, b and c for each segment are outlined in Table 2. For the sixth

TABLE 2: ERROR FUNCTION APPROXIMATION POLYNOMIAL COEFFICIENTS

segment	a	b	c
1: 0.0-0.5	-0.2539	1.1755	-0.0018
2: 0.5-1.0	-0.4739	1.3547	-0.0384
3: 1.0-1.5	-0.2959	0.9828	0.1568
4: 1.5-2.0	-0.0950	0.3888	0.5972
5: 2.0-3.0	-0.0075	0.0409	0.9440
6: 3.0-∞	0.0	0.0	1.0

segment, the value of the error function is set to unity, since by constraint the error function is very close to a value of one for an argument value beyond 3.0. The comparison of the actual error function to its approximations, along with the corresponding residuals, are presented in Figures 5 to 14. Note that smaller residuals can be obtained by approximating with a higher order of polynomial. A conditional statement is included inside the integration loop to verify if the error function argument is negative. This is in order to limit

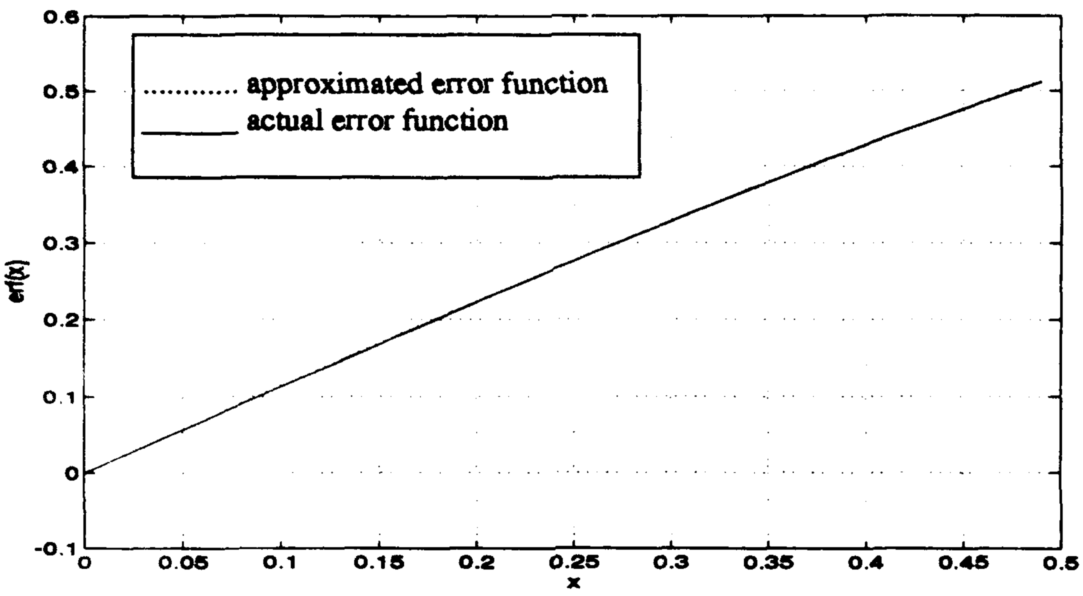

Figure 5: Segment 1 of Error Function Approximation

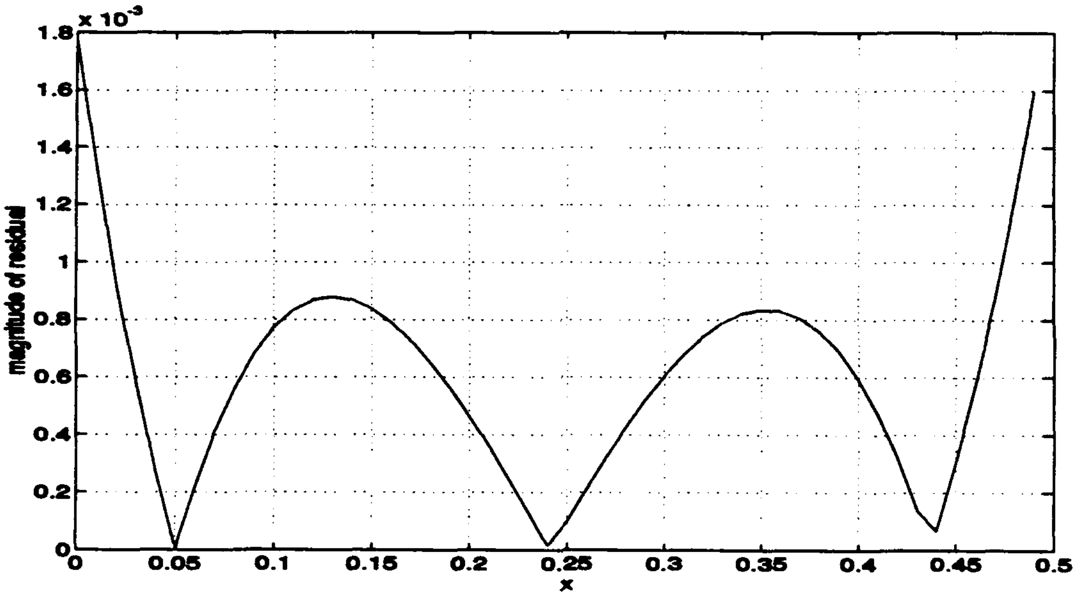

Figure 6: Residual of Segment 1 of Error Function Approximation

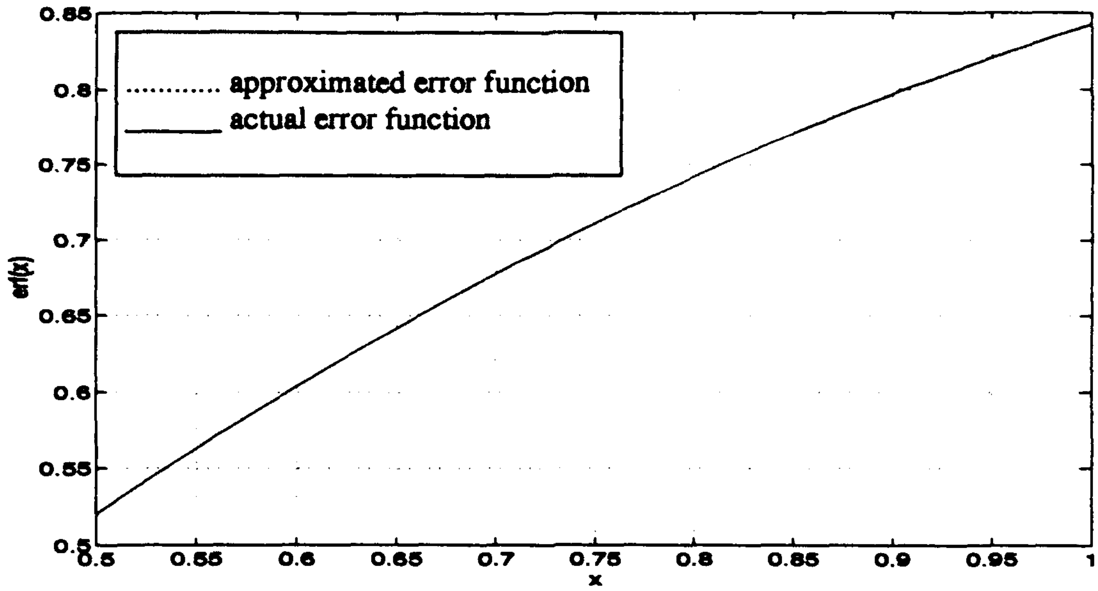

Figure 7: Segment 2 of Error Function Approximation

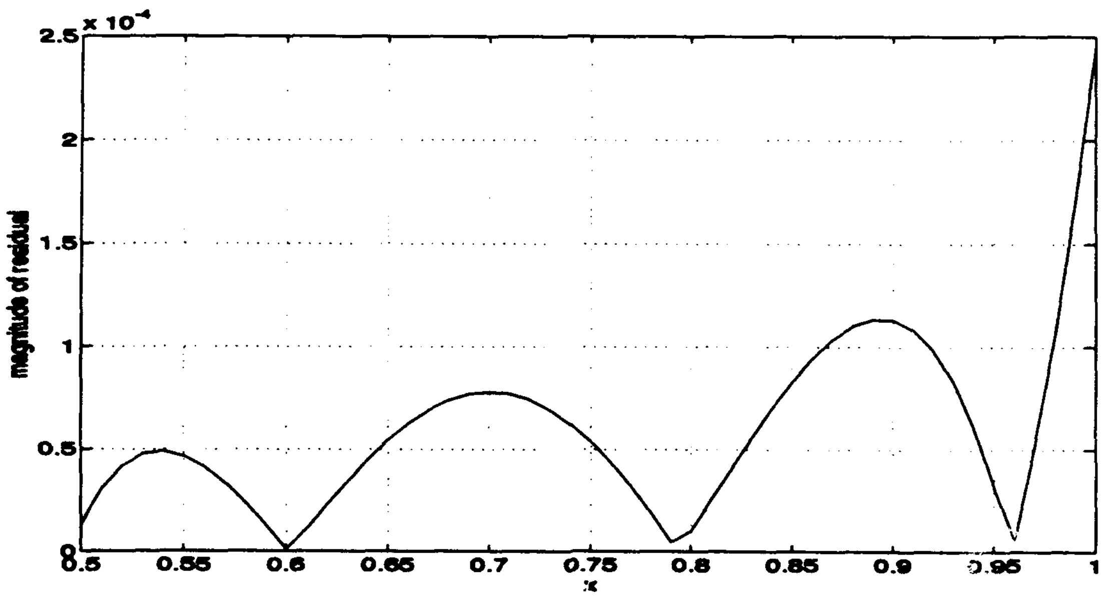

Figure 8: Residual of Segment 2 of Error Function Approximation

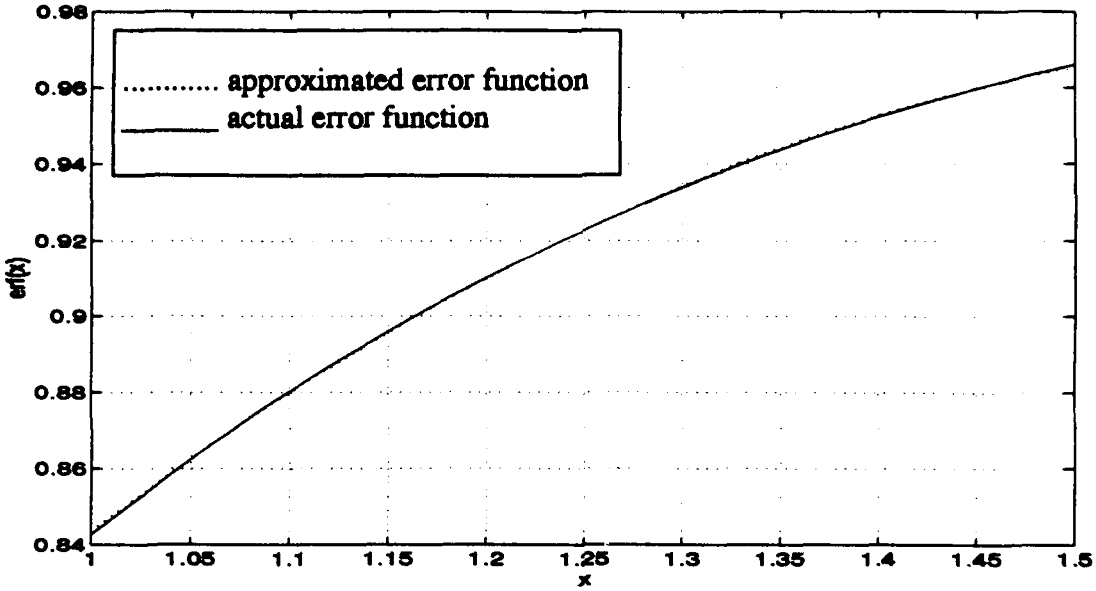

Figure 9: Segment 3 of Error Function Approximation

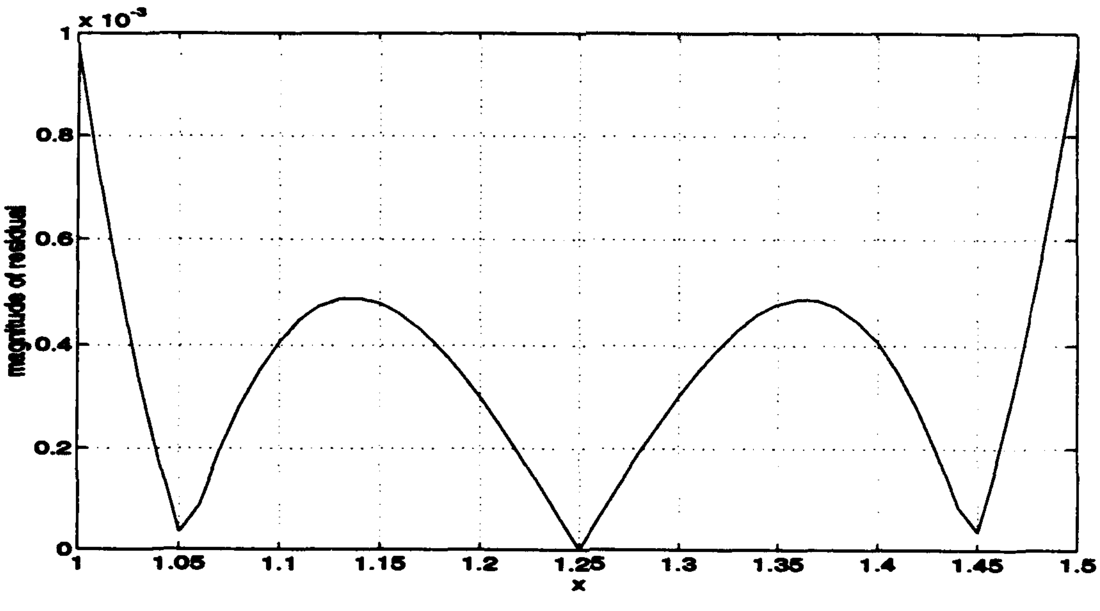

Figure 10: Residual of Segment 3 of Error Function Approximation

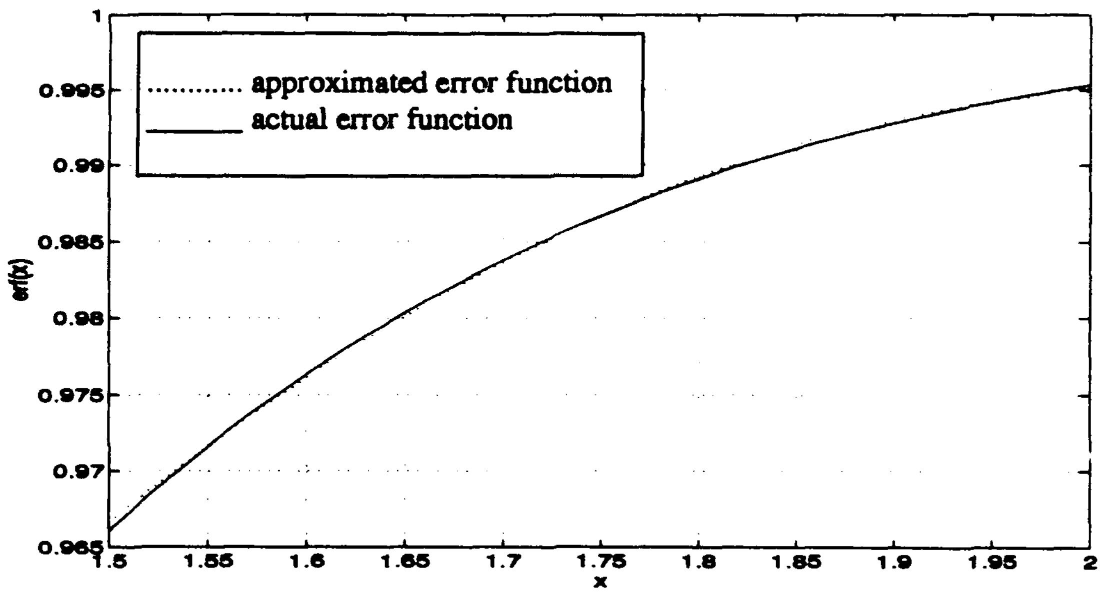

Figure 11: Segment 4 of Error Function Approximation

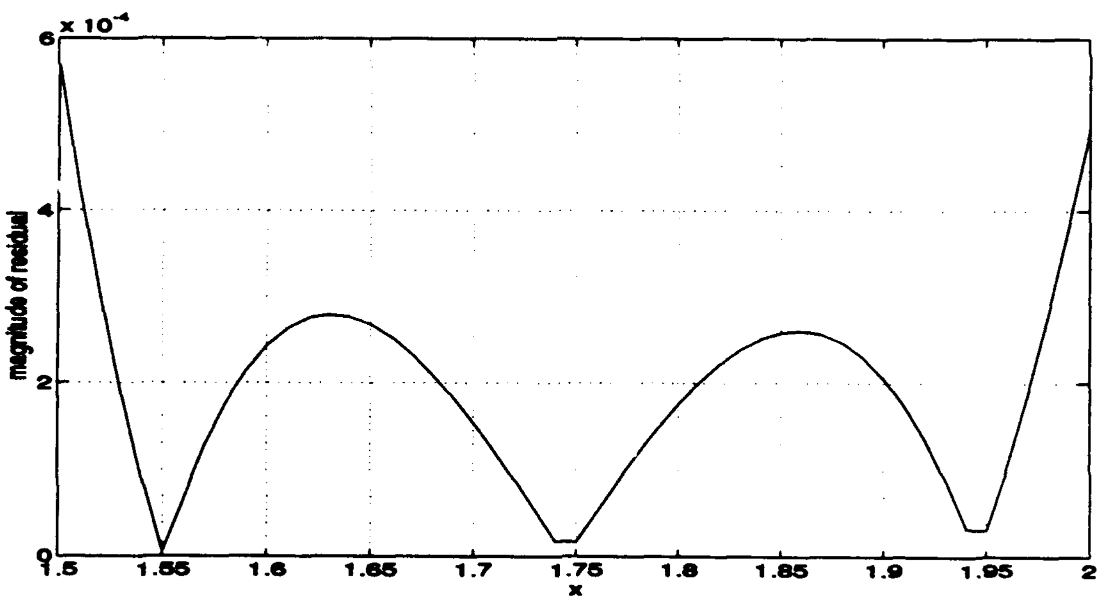

Figure 12: Residual of Segment 4 of Error Function Approximation

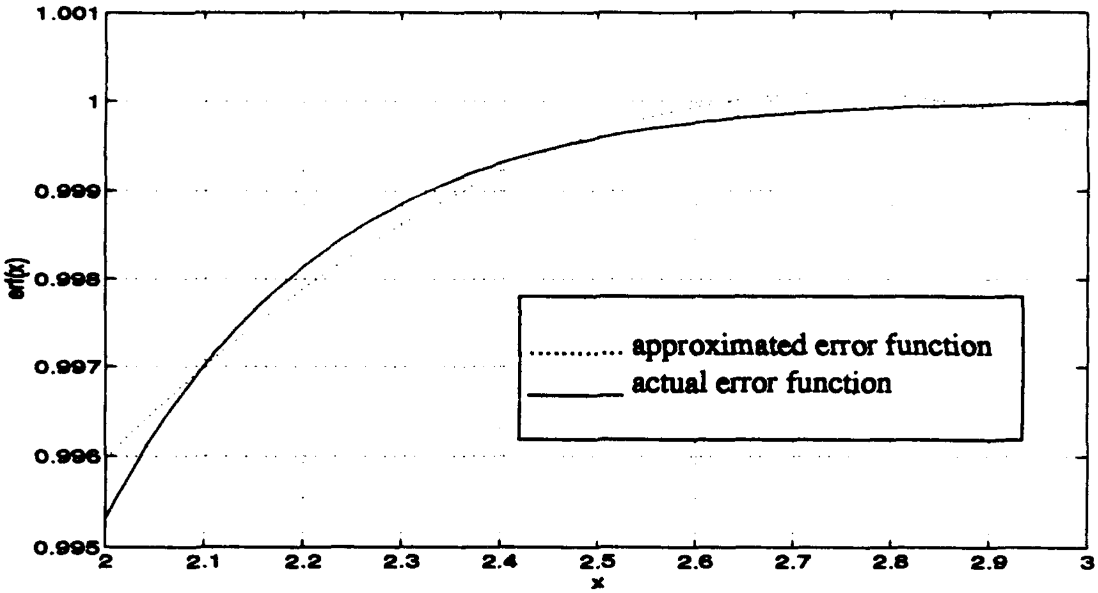

Figure 13: Segment 5 of Error Function Approximation

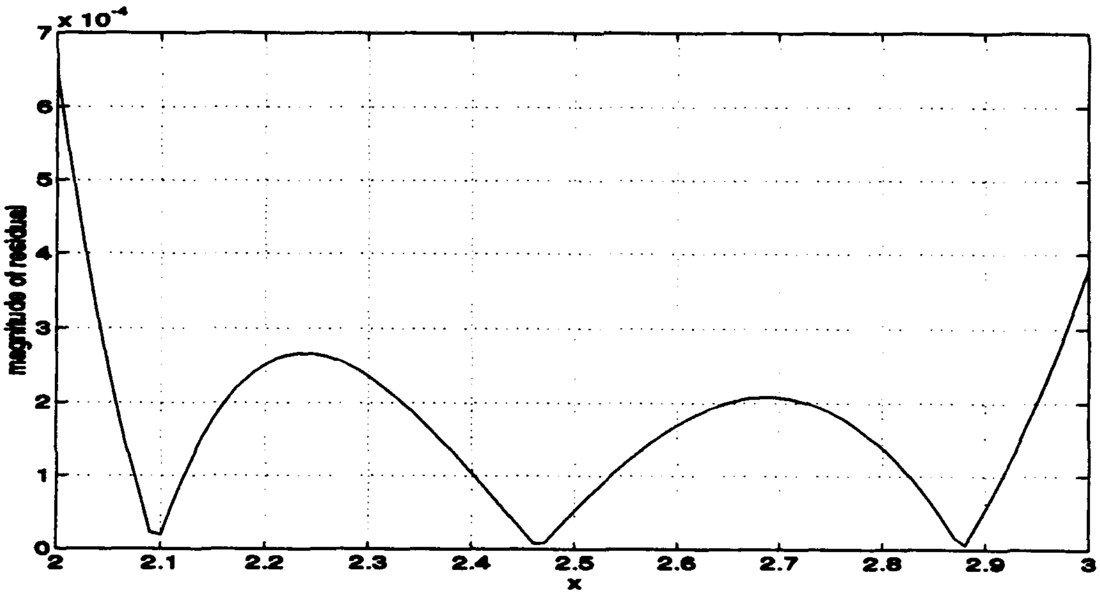

Figure 14: Residual of Segment 5 of Error Function Approximation

the number of additional integration terms introduced by the error function approximation. If the argument is negative, the value of the error function is found from the relation $erf(-x) = -erf(x)$ [Ref. 9]. These curves are generated using the program in Appendix A.

As an example of how to integrate (52) using the error function approximations, the first term (-2EQ) will be integrated here, for the range where the argument of the error function of E (57) is between 0.5 and 1.0 (segment 2). In other words, when the integration of (52) (integrating over u) is being performed between the limits $u = a(m_Q - 1/\sqrt{2})$ and $u = a(m_Q - \sqrt{2})$, integration of the first term (referred to as $I_1(z)$) is expressed as

$$I_1(z) = \int_{a(m_Q-1/\sqrt{2})}^{a(m_Q-\sqrt{2})} -2EQe^{\frac{-1}{2a^2b^2}(-2zua^2+u^2b^2+3m_Q^2a^2b^2+z^2a^2+u^2a^2+2um_Qa^2b+2zm_Q^2a^2b+2zm_ta^2b+3m_t^2a^2b^2+2um_ta^2b+2um_tab^2+2um_Qab^2)}du$$

$$= -2\int_{a(m_Q-1/\sqrt{2})}^{a(m_Q-\sqrt{2})} \left\{ e^{-0.4739\left(\frac{-u+m_Qa}{a\sqrt{2}}\right)^2+1.3547\left(\frac{-u+m_Qa}{a\sqrt{2}}\right)-0.0384} \times \right.$$

$$e^{\frac{1}{ab}(2zm_Qa+m_Q^2ab+zm_ta+um_ta+m_t^2ab+um_Qb)} \times$$

$$e^{\frac{-1}{2a^2b^2}(-2zua^2+u^2b^2+3m_Q^2a^2b^2+z^2a^2+u^2a^2+2um_Qa^2b+2zm_Q^2a^2b+2zm_ta^2b+3m_t^2a^2b^2+2um_ta^2b+2um_tab^2+2um_Qab^2)} \left.\vphantom{\int}\right\} du$$

or

$$I_1(z) = \left[\frac{0.00005}{(m_I a + m_Q b)^3} e^{\frac{1}{b}(2zm_Q + 2m_Q^2 b + zm_I + m_I a m_Q - 1.4142 m_I a + m_I^2 b - 1.4142 m_Q b)} \times \right.$$

$$ba\,(16848 m_Q^2 b^2 + 16848 m_I^2 a^2 + 5754.435 abm_I - 9478 b^2 + 33696 m_I a m_Q b +$$

$$\left. 5754.435 b^2 m_Q) \right] -$$

$$\left[\frac{0.000025}{(m_I a + m_Q b)^3} e^{\frac{1}{2b}(4zm_Q + 4m_Q^2 b + 2zm_I + 2m_I a m_Q - 1.4142 m_I a + 2m_I^2 b - 1.4142 m_Q b)} \times \right.$$

$$ba\,(20819 m_Q^2 b^2 + 20819 m_I^2 a^2 + 24912.786 abm_I - 18956 b^2 + 41638 m_I a m_Q b +$$

$$\left. 24912.786 b^2 m_Q) \right] \qquad (80)$$

Using this error function approximation in (52) would result in a closed form target test cell density function $p_c(x)$ being composed of over 1100 exponential terms. Hence, this may be a very costly method of determining $p_c(x)$ for a given situation.

B. CLOSED FORM PDF BY CURVE-FITTING

The probability density function obtained in Section III.A.6 for the envelope detection approximation $x_e = aMax\{|I|, |Q|\} + bMin\{|I|, |Q|\}$ cannot be practically implemented for either system performance analysis or actual operational use. This is due to the very complex and tedious procedure required to convert the convolution integral of the test cell (52) into a closed form expression. Of course, the integral could be evaluated numerically, but this approach would be extremely computationally intensive and time-

consuming. This is especially true for the computation of probability of detection curves for the GO CFAR processor, which require at least four integrations (see Section III.A).

The alternative presented here provides a relatively simple closed form expression for the density function of the target test cell with arbitrary SNR. Generally, the closed form solution for a target test cell density function can be modelled as an exponential of the form

$$p_c(x) \approx e^{Ax^{10}+Bx^9+Cx^8+Dx^7+Ex^6+Fx^5+Gx^4+Hx^3+Ix^2+Jx+K} \tag{81}$$

where the coefficients $A, B, ..., K$ are functions of the SNR and may be expressed as polynomials of the form

$$A = A_8s^8+A_7s^7+A_6s^6+A_5s^5+A_4s^4+A_3s^3+A_2s^2+A_1s+A_0 \tag{82}$$

with similar expansions for $B, ..., K$. The variable s represents the SNR in decibels.

Polynomial curve-fitting in a least-squares sense is used to yield (81). Such curve-fitting, using a tenth order exponential as shown above, generally provides a maximum residual error magnitude of 0.03 if the range of the density function over which the curve-fitting is applied is small enough. Now, as is well known, the constraint of "small enough" is ambiguous. In this research, the range over which the density function is greater than 10^{-5} was considered the limit of the curve-fitting range. A tenth order exponential is used in (81) so as to maintain a minimum residual error. These, however, have the disadvantage of not being integrable in closed form, and have to be integrated numerically. Higher order exponentials provided no further improvement in the residual error. Smaller order exponentials in (81) would ensure that a closed form solution could be obtained for the integration of the target test cell density function $p_c(x)$ in (7). Such smaller order exponentials, however, proved to be very inaccurate.

The solution of the form (81) is obtained by performing the numerical convolution (50) for a number of increasing signal-to-noise ratios and for a given combination of the multiplying constants a and b of the envelope detection approximation. Each numerical convolution was averaged over the phase angle in the range $0 \ldots \pi/4$, as described in Section II.B. The results of such an operation are shown in Figure 15, where the

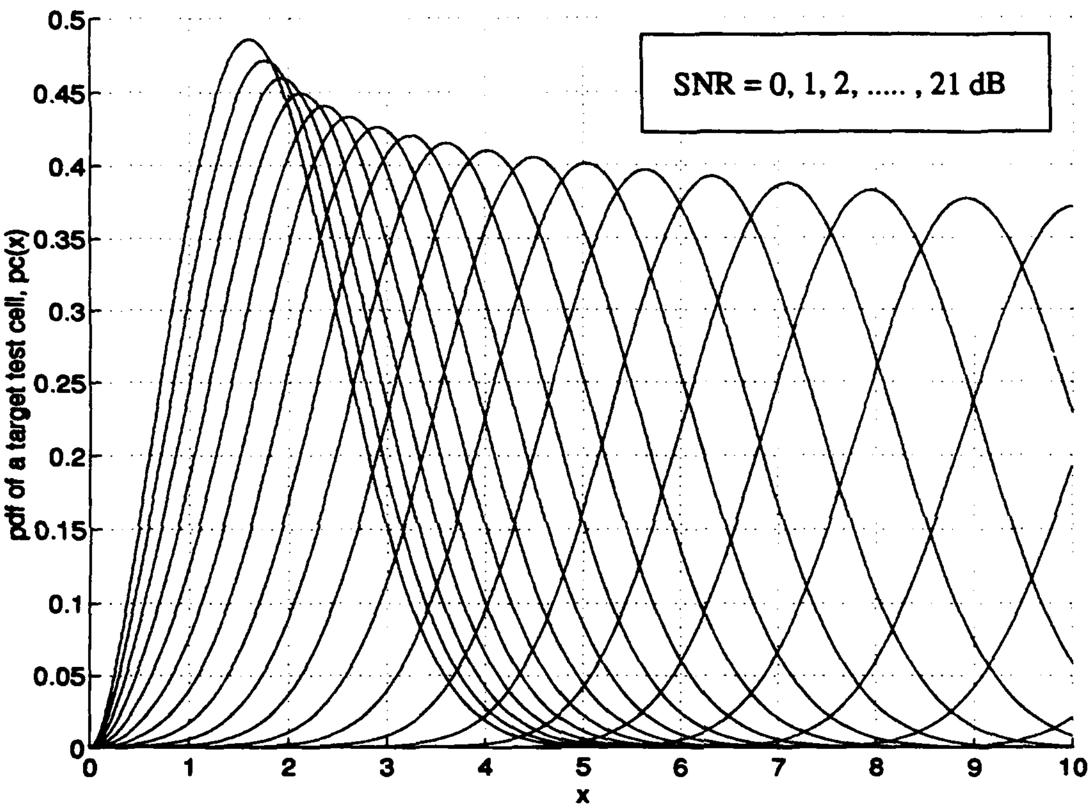

Figure 15: Test Cell PDF for Increasing SNR

consecutive density function curves correspond to the test cell density p_c for each of the signal-to-noise ratios (SNRs) used and where the constants were chosen to be a=1.0 and b=0.25. These particular constant values were chosen arbitrarily in order to demonstrate the present discussion. These curves are generated using the program in Appendix B.

Each curve of Figure 15 is then curve-fitted, independently of the others, in the form of (81). For example, modeling of the first curve (SNR = 0 dB) over the range $x = 0.01 \ldots 5.6$ (region where $p_c > 10^{-5}$) yields the curve-fit shown in Figure 16, which

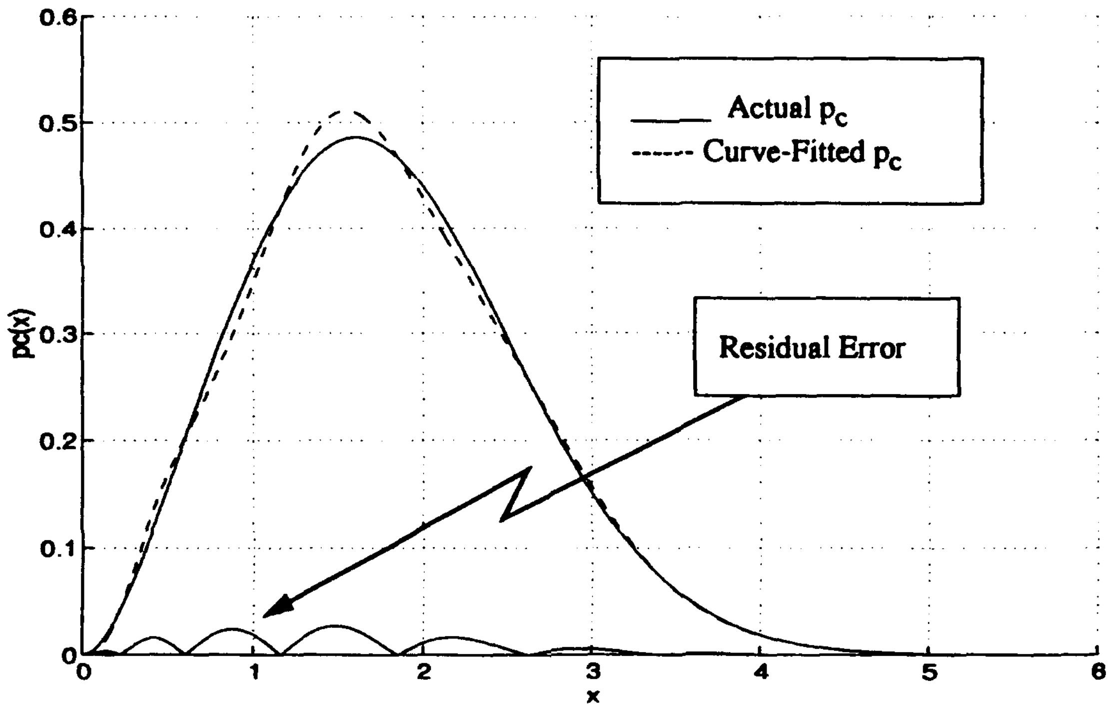

Figure 16: Curve-Fit of $p_c(x)$ for SNR = 0 dB, $a = 1.0$, $b = 0.25$

also shows the corresponding magnitude of the residual. The exponential coefficients required for this curve-fit are

$$A = -0.0015$$

$$B = 0.0459$$

$$C = -0.5889$$

$$D = 4.262$$

$$E = -19.1293$$

$$F = 55.1303$$

$$G = -102.2169$$

$$H = 119.0197$$

$$I = -83.2403$$

$$J = 33.1493$$

$$K = -7.482$$

Outside of the curve-fit range, the value of $p_c(x)$ is taken to be zero.

By performing a number of these curve-fits for successive $p_c(x)$ curves in Figure 15, sequences of coefficients for each of $A,B, \ldots\ldots, K$ can be plotted out. All the curve-fits thus obtained are shown in Figure 17, with their corresponding residual error magnitudes. The coefficient sequences are plotted out for SNR = 0....21 dB and for $a = 1.0$ and $b = 0.25$ in Figures 18 to 28. It is rather difficult to extract valid coefficient values from these plots. Therefore the exact coefficient values used to produce the plots are tabulated. This is actually done for all the envelope detection approximation situations outlined in Table 1 so as to provide a good reference design tool. One density $p_c(x)$ can be produced at each integer value of SNR. Tables 3 to 9 provide all the necessary coefficients. The SNR range 0....21 dB was chosen since it covers the range over which P_D is less than one (see section V). Beyond this range, P_D is constant at one.

The curves for $A,B, \ldots\ldots, K$ can in turn be curve-fitted in the form of (82). For instance, the exponential coefficient A of (81) can be expressed as (82) where the coefficients A_8, $A_7, \ldots\ldots, A_0$ are determined by the polynomial curve-fit. Thus, for a given a and b and SNR range, it appears that the density function of a target test cell could be approximated by both (81) and (82). This would, in effect, remove the dependence of (81) on the SNR. The coefficient curves are also compared to their curve-fits in Figures 18 to 28. As well, these

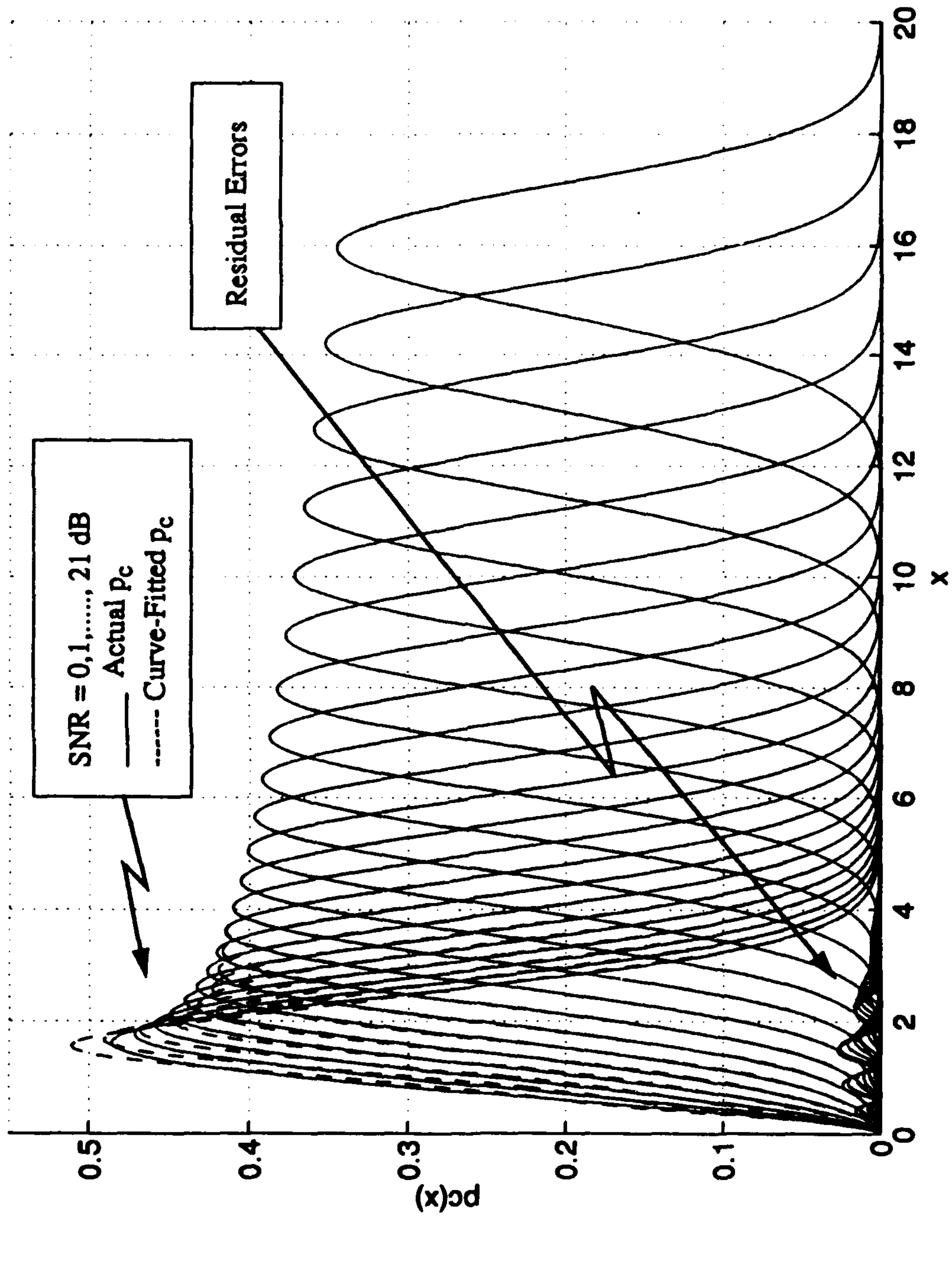

Figure 17: Curve-Fits for $p_c(x)$, with Residual Errors

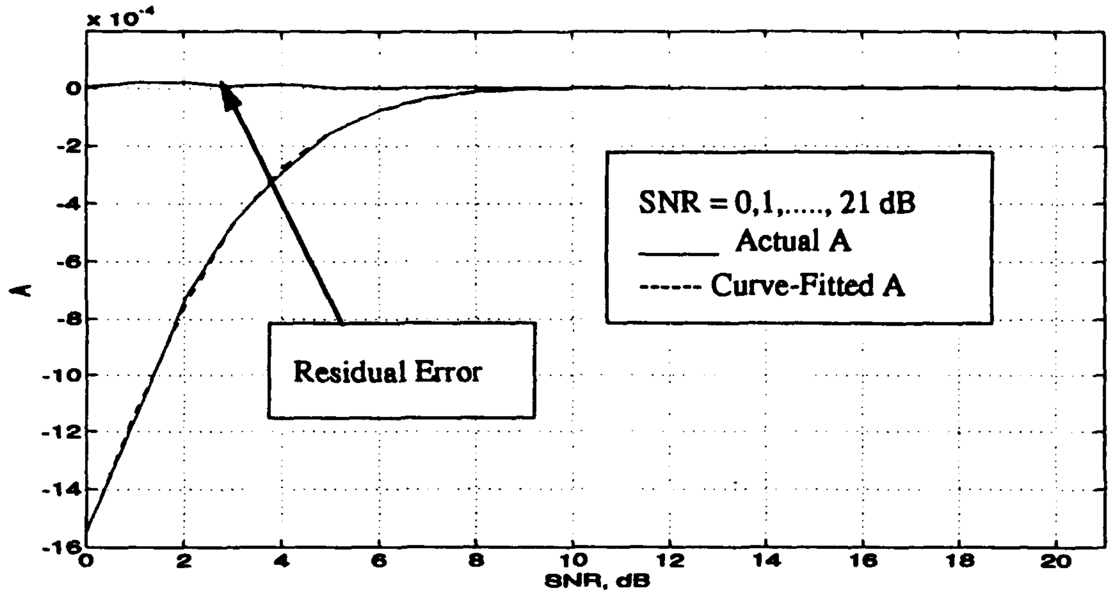

Figure 18: Behaviour of Exponential Coefficient A as a Function of SNR

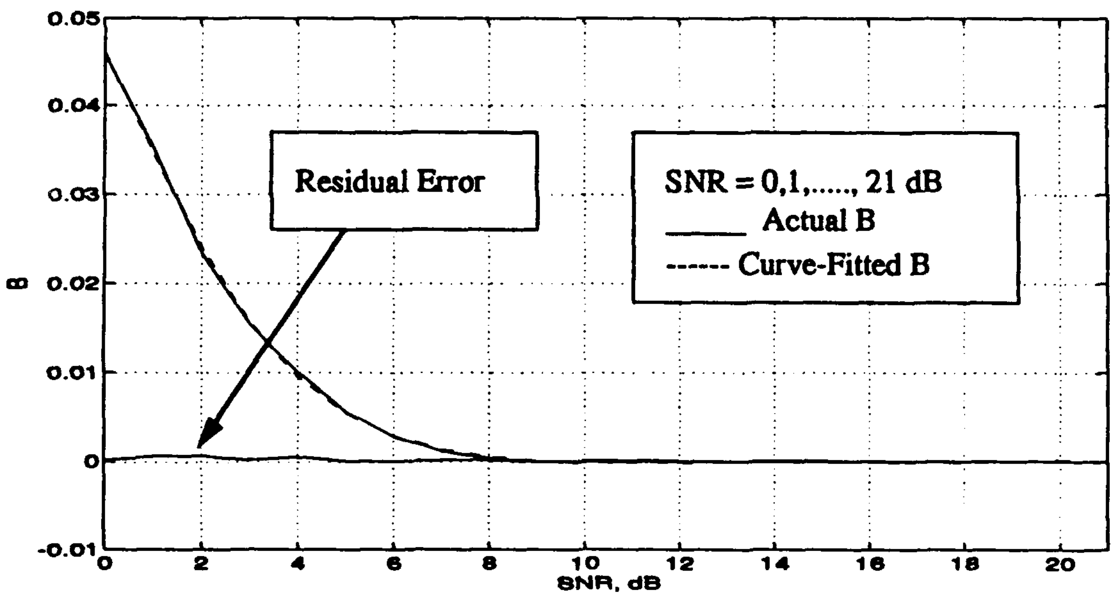

Figure 19: Behaviour of Exponential Coefficient B as a Function of SNR

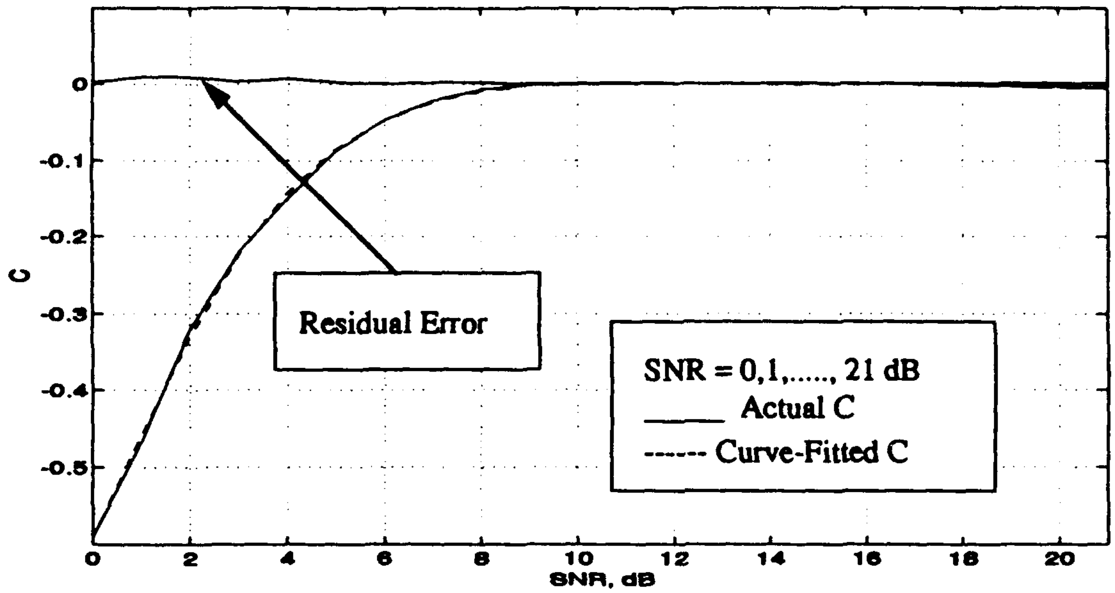

Figure 20: Behaviour of Exponential Coefficient C as a Function of SNR

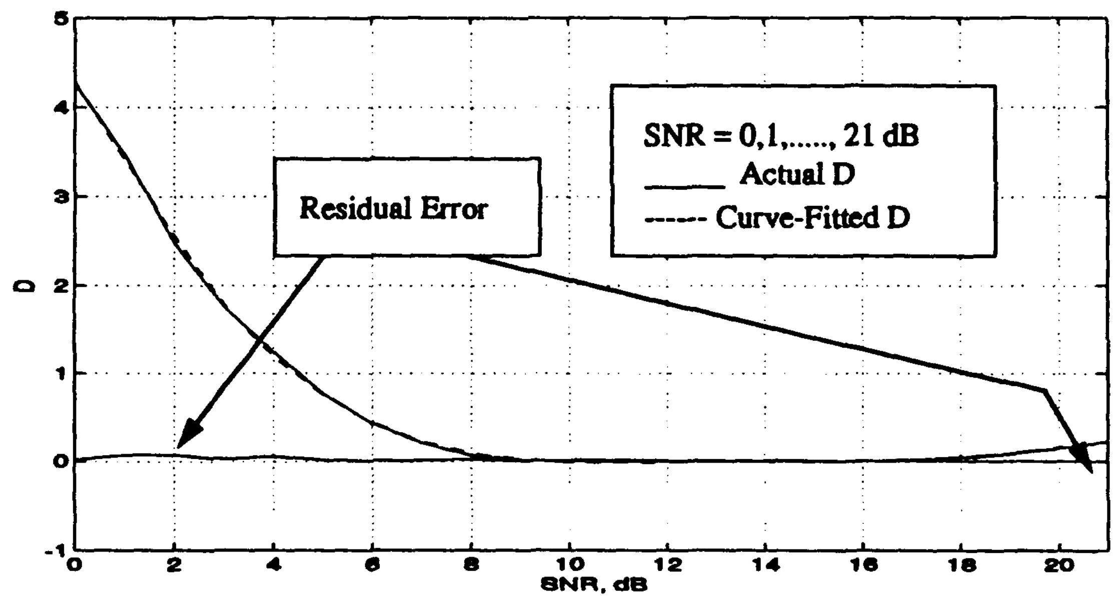

Figure 21: Behaviour of Exponential Coefficient D as a Function of SNR

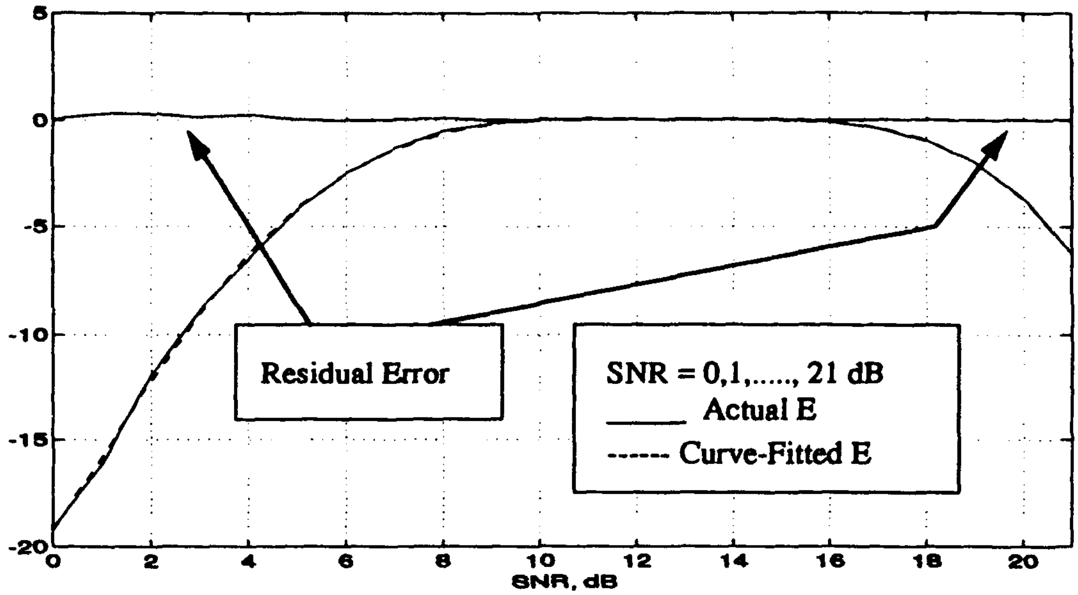

Figure 22: Behaviour of Exponential Coefficient E as a Function of SNR

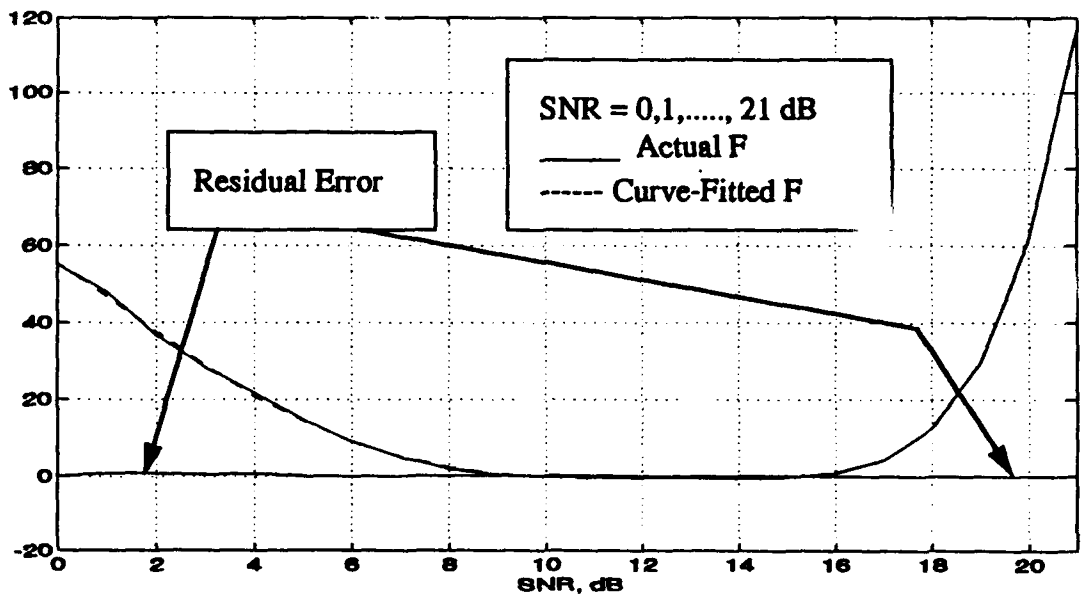

Figure 23: Behaviour of Exponential Coefficient F as a Function of SNR

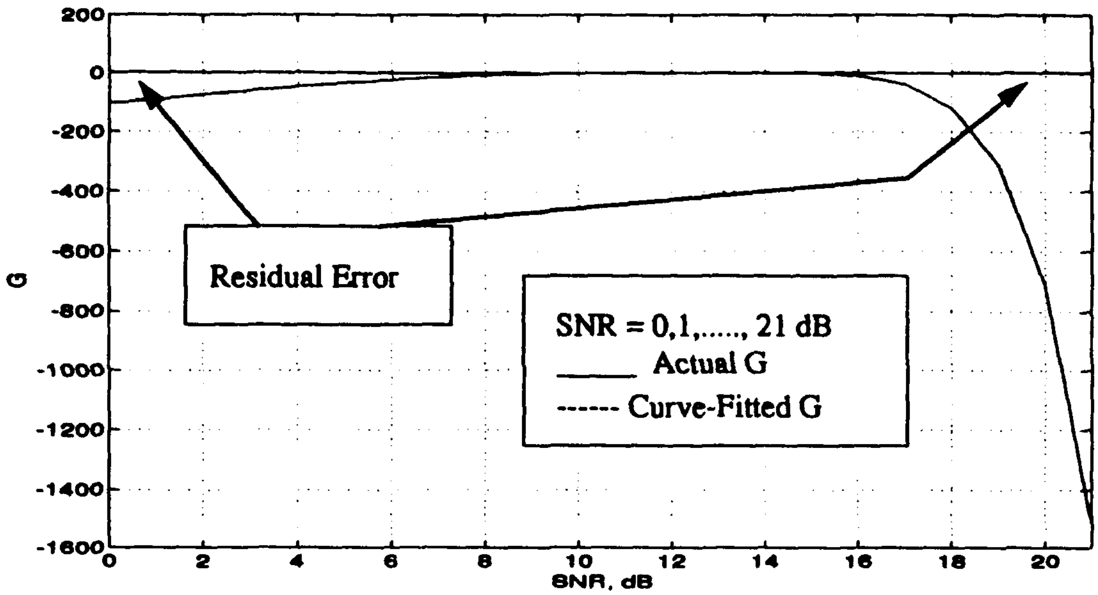

Figure 24: Behaviour of Exponential Coefficient G as a Function of SNR

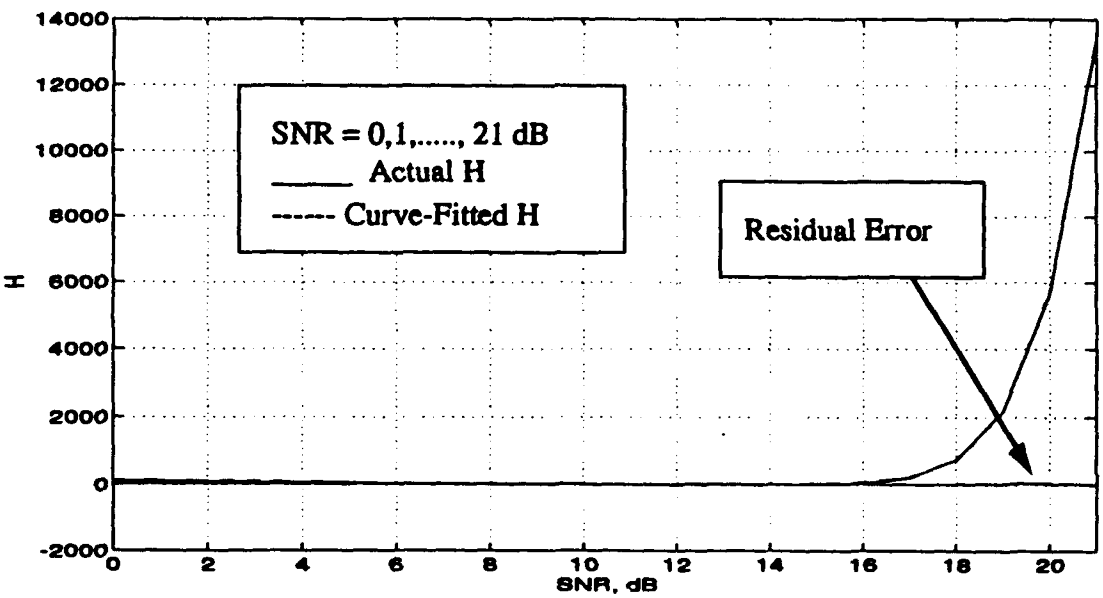

Figure 25: Behaviour of Exponential Coefficient H as a Function of SNR

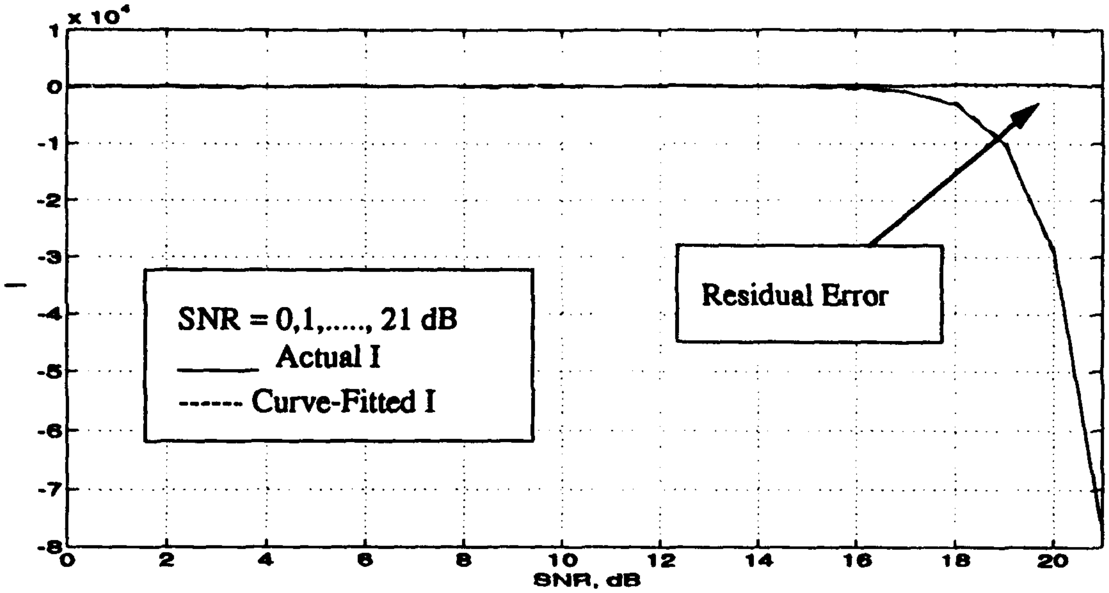

Figure 26: Behaviour of Exponential Coefficient I as a Function of SNR

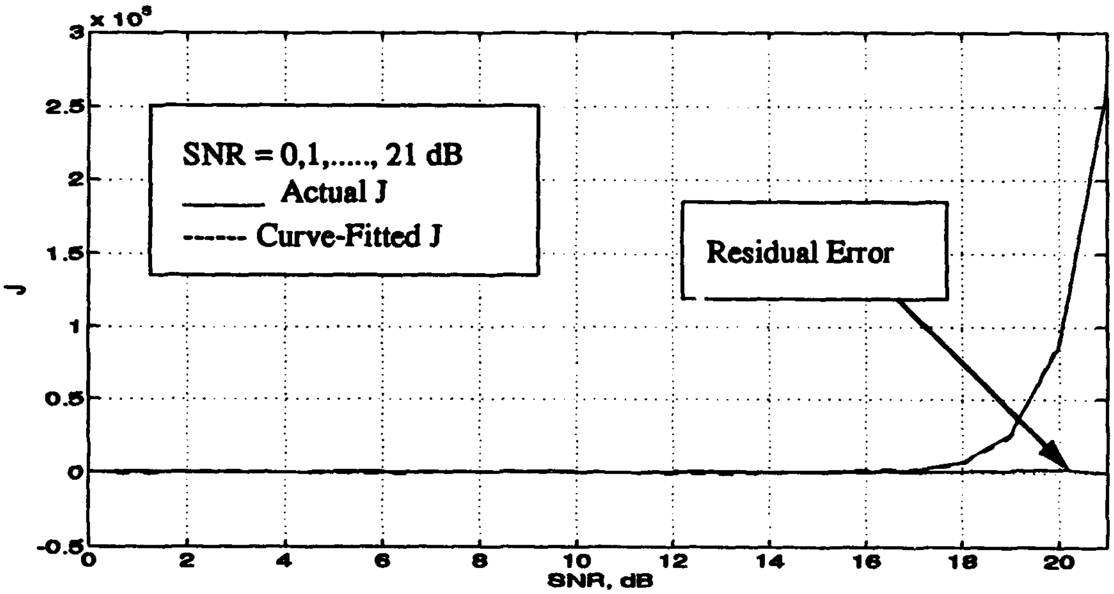

Figure 27: Behaviour of Exponential Coefficient J as a Function of SNR

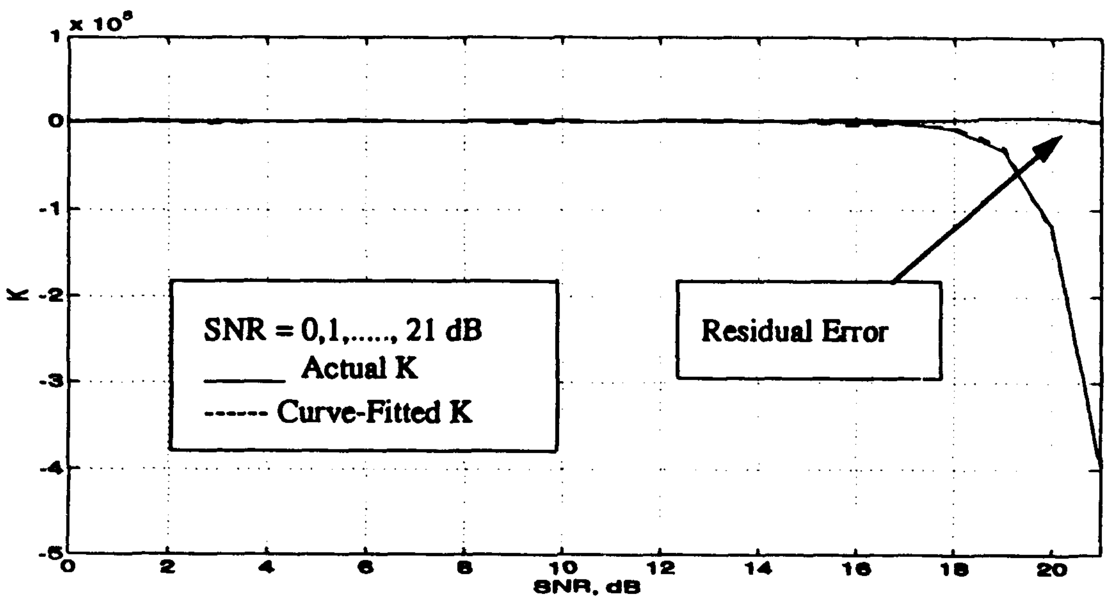

Figure 28: Behaviour of Exponential Coefficient K as a Function of SNR

show the corresponding residual error magnitudes. Note that the residual error in each case hovers very near zero, indicating an effective curve-fit. As it turns out, however, attempts to obtain a reasonable target test cell density function using the curve-fits of the coefficients $A, B, \ldots\ldots, K$ were unsuccessful. It appears that the result of a polynomial expansion as in (82) is very sensitive to slight changes in its coefficient values. Hence, since a curve-fit generally has errors at every point, regardless of how small, the resultant polynomial does not behave as expected.

As mentioned earlier, smaller order exponential expansions approximating $p_c(x)$ generate significant errors, as can be seen in Figure 29, which shows a third order exponential curve-fit.

The program used to generate all the curve-fits and plots in this discussion is in Appendix C.

TABLE 3: EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 1.0$, FOR x_e

SNR (dB)	Range Lower Limit x_L	Range Upper Limit x_U	A	B	C
0	.02	6.78	-1.9238e-4	6.9423e-3	-1.0802e-1
1	.025	7.02	-1.2619e-4	4.7207e-3	-7.6157e-2
2	.03	7.295	-8.0869e-5	3.1468e-3	-5.2814e-2
3	.04	7.6	-4.7671e-5	1.9360e-3	-3.3925e-2
4	.055	7.945	-2.6193e-5	1.1146e-3	-2.0473e-2
5	.08	8.33	-1.3038e-5	5.8365e-4	-1.1284e-2
6	.125	8.765	-5.5545e-6	2.6303e-4	-5.3839e-3
7	.22	9.25	-1.7716e-6	8.9455e-5	-1.9557e-3
8	.4	9.8	-3.9263e-7	2.1412e-5	-5.0709e-4
9	.705	10.415	-5.3040e-8	3.2497e-6	-8.7064e-5
10	1.11	11.105	2.3166e-8	-1.2966e-6	3.0154e-5
11	1.61	11.88	4.6389e-8	-2.9893e-6	8.2487e-5
12	2.2	12.745	4.9775e-8	-3.5120e-6	1.0648e-4
13	2.89	13.72	4.5677e-8	-3.5275e-6	1.1728e-4
14	3.68	14.81	4.0119e-8	-3.4077e-6	1.2490e-4
15	4.58	16.035	3.1905e-8	-2.9665e-6	1.1892e-4
16	5.59	17.405	2.1449e-8	-2.1390e-6	9.0897e-5
17	6.715	18.945	1.0642e-8	-1.0544e-6	4.1129e-5
18	7.97	20.67	1.8897e-9	3.6074e-9	-1.8714e-5
19	9.375	22.605	-3.8683e-9	8.4043e-7	-7.6459e-5
20	10.945	24.775	-6.3980e-9	1.3038e-6	-1.1726e-4
21	12.7	27.21	-6.5285e-9	1.4038e-6	-1.3443e-4

TABLE 3 (cont'd): EXPONENTIAL COEFFICIENTS, a = 1.0, b = 1.0, FOR x_e

SNR (dB)	D	E	F	G	H
0	2.000e-2	6.7800e+0	-1.9238e-4	6.9423e-3	-1.0802e-1
1	2.500e-2	200e+0	-1.2619e-4	4.7207e-3	-7.6157e-2
2	3.000e-2	7.2950e+0	-8.0869e-5	3.1468e-3	-5.2814e-2
3	4.000e-2	7.6000e+0	-4.7671e-5	1.9360e-3	-3.3925e-2
4	5.500e-2	7.9450e+0	-2.6193e-5	1.1146e-3	-2.0473e-2
5	8.000e-2	8.3300e+0	-1.3038e-5	5.8365e-4	-1.1284e-2
6	1.250e-1	8.7650e+0	-5.5545e-6	2.6303e-4	-5.3839e-3
7	2.200e-1	9.2500e+0	-1.7716e-6	8.9455e-5	-1.9557e-3
8	4.000e-1	9.8000e+0	-3.9263e-7	2.1412e-5	-5.0709e-4
9	7.050e-1	1.0415e+1	-5.3040e-8	3.2497e-6	-8.7064e-5
10	1.1100e+0	1.1105e+1	2.3166e-8	-1.2966e-6	3.0154e-5
11	1.6100e+0	1.1880e+1	4.6389e-8	-2.9893e-6	8.2487e-5
12	2.2000e+0	1.2745e+1	4.9775e-8	-3.5120e-6	1.0648e-4
13	2.8900e+0	1.3720e+1	4.5677e-8	-3.5275e-6	1.1728e-4
14	3.6800e+0	1.4810e+1	4.0119e-8	-3.4077e-6	1.2490e-4
15	4.5800e+0	1.6035e+1	3.1905e-8	-2.9665e-6	1.1892e-4
16	5.5900e+0	1.7405e+1	2.1449e-8	-2.1390e-6	9.0897e-5
17	6.7150e+0	1.8945e+1	1.0642e-8	-1.0544e-6	4.1129e-5
18	7.9700e+0	2.0670e+1	1.8897e-[illegible]	3.6074e-9	-1.8714e-5
19	9.3750e+0	2.2605e+1	-3.8683e-9	8.4043e-7	-7.6459e-5
20	1.0945e+1	2.4775e+1	-6.3980e-9	1.3038e-6	-1.1726e-4
21	1.2700e+1	2.7210e+1	-6.5285e-9	1.4038e-6	-1.3443e-4

TABLE 3 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 1.0$, FOR x_e

SNR (dB)	I	J	K
0	-5.000e+1	2.5681e+1	-8.277e+0
1	-4.434e+1	2.4021e+1	-8.463e+0
2	-3.929e+1	2.2521e+1	-8.740e+0
3	-3.330e+1	2.0556e+1	-9.041e+0
4	-2.733e+1	1.8463e+1	-9.418e+0
5	-2.129e+1	1.6166e+1	-9.875e+0
6	-1.528e+1	1.3658e+1	-1.042e+1
7	-9.420e+0	1.0912e+1	-1.106e+1
8	-5.048e+0	8.6388e+0	-1.196e+1
9	-2.679e+0	7.3700e+0	-1.332e+1
10	-1.359e+0	6.7302e+0	-1.520e+1
11	-1.7187e-2	5.9774e+0	-1.748e+1
12	1.4997e+0	5.1285e+0	-2.041e+1
13	3.4151e+0	4.4415e+0	-2.482e+1
14	6.7708e+0	3.8751e+0	-3.266e+1
15	3.2824e+0	2.4171e+1	-6.785e+1
16	-4.670e+1	1.7287e+2	-2.608e+2
17	-2.556e+2	8.0545e+2	-1.105e+3
18	-8.599e+2	2.8178e+3	-4.058e+3
19	-2.293e+3	8.1620e+3	-1.284e+4
20	-5.103e+3	2.0022e+4	-3.484e+4
21	-9.761e+3	4.2408e+4	-8.194e+4

TABLE 4: EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 0.5$, FOR x_e

SNR (dB)	Range Lower Limit x_L	Range Upper Limit x_U	A	B	C
0	.015	5.89	-8.2002e-4	2.5679e-2	-3.4668e-1
1	.015	6.085	-5.9881e-4	1.9372e-2	-2.7018e-1
2	.02	6.3	-3.8511e-4	1.2918e-2	-1.8686e-1
3	.03	6.545	-2.2335e-4	7.8035e-3	-1.1764e-1
4	.04	6.82	-1.3015e-4	4.7479e-3	-7.4765e-2
5	.055	7.125	-7.1127e-5	2.7179e-3	-4.4848e-2
6	.09	7.47	-3.1612e-5	1.2734e-3	-2.2174e-2
7	.155	7.86	-1.1488e-5	4.9103e-4	-9.0876e-3
8	.3	8.29	-2.7631e-6	1.2652e-4	-2.5165e-3
9	.57	8.775	-4.8343e-7	2.3765e-5	-5.0945e-4
10	.955	9.32	-1.1659e-7	5.9043e-6	-1.2978e-4
11	1.44	9.93	-3.4433e-8	1.6178e-6	-3.0537e-5
12	2.02	10.615	-1.1632e-8	3.8520e-7	1.3429e-7
13	2.69	11.38	-1.0728e-8	4.5402e-7	-3.8989e-6
14	3.465	12.235	-1.3047e-8	7.5926e-7	-1.6489e-5
15	4.35	13.2	-1.0466e-8	6.6902e-7	-1.5785e-5
16	5.345	14.275	-5.4726e-9	3.0051e-7	-2.3702e-6
17	6.47	15.485	9.112e-10	-3.1705e-7	2.6409e-5
18	7.73	16.845	7.4178e-9	-1.0975e-6	7.0533e-5
19	9.145	18.365	1.2680e-8	-1.8827e-6	1.2429e-4
20	10.720	20.075	1.4164e-8	-2.2461e-6	1.5917e-4
21	12.485	21.99	9.0439e-9	-1.5160e-6	1.1354e-4

TABLE 4 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 0.5$, FOR x_e

SNR (dB)	D	E	F	G	H
0	2.6410e+0	-1.248e+1	3.7896e+1	-7.414e+1	9.1405e+1
1	2.1262e+0	-1.038e+1	3.2549e+1	-6.575e+1	8.3637e+1
2	1.5258e+0		2.5194e+1	-5.293e+1	7.0114e+1
3	1.0018e+0	-5.299e+0	1.8047e+1	-3.970e+1	5.5211e+1
4	6.6539e-1	-3.681e+0	1.3122e+1	-3.025e+1	4.4157e+1
5	4.1852e-1	-2.430e+0	9.0994e+0	-2.207e+1	3.3975e+1
6	2.1867e-1	-1.344e+0	5.3428e+0	-1.381e+1	2.2771e+1
7	9.5451e-2	-6.2675e-1	2.6724e+0	-7.451e+0	1.3385e+1
8	2.8529e-2	-2.0342e-1	9.5011e-1	-2.939e+0	5.9730e+0
9	6.2662e-3	-4.9004e-2	2.5533e-1	-9.0354e-1	2.1830e+0
10	1.6361e-3	-1.3242e-2	7.3431e-2	-2.9130e-1	8.5036e-1
11	2.8751e-4	-1.3054e-3	1.7862e-3	5.5201e-4	6.7262e-2
12	-1.8672e-4	3.7803e-3	-3.6964e-2	2.0507e-1	-6.4428e-1
13	-1.2208e-4	3.6073e-3	-4.3359e-2	2.9147e-1	-1.150e+0
14	1.2564e-4	1.0598e-3	-3.1541e-2	3.0413e-1	-1.580e+0
15	1.1806e-4	1.8113e-3	-5.1991e-2	5.6644e-1	-3.452e+0
16	-2.2027e-4	8.3078e-3	-1.4429e-1	1.4839e+0	-9.526e+0
17	-1.0750e-3	2.5944e-2	-4.0165e-1	4.1088e+0	-2.769e+1
18	-2.6083e-3	6.1696e-2	-9.7786e-1	1.0538e+1	-7.635e+1
19	-4.8051e-3	1.2050e-1	-2.049e+0	2.3915e+1	-1.893e+2
20	-6.6375e-3	1.8039e-1	-3.339e+0	4.2635e+1	-3.709e+2
21	-5.0050e-3	1.4385e-1	-2.818e+0	3.8142e+1	-3.524e+2

TABLE 4 (cont'd): EXPONENTIAL COEFFICIENTS, a = 1.0, b = 0.5, FOR x_e

SNR (dB)	I	J	K
0	-6.837e+1	3.0021e+1	-7.912e+0
1	-6.447e+1	2.9281e+1	-8.219e+0
2	-5.646e+1	2.7182e+1	-8.458e+0
3	-4.696e+1	2.4447e+1	-8.708e+0
4	-3.961e+1	2.2285e+1	-9.119e+0
5	-3.235e+1	1.9990e+1	-9.631e+0
6	-2.360e+1	1.6866e+1	-1.016e+1
7	-1.553e+1	1.3674e+1	-1.083e+1
8	-8.343e+0	1.0434e+1	-1.164e+1
9	-4.097e+0	8.3157e+0	-1.289e+1
10	-2.318e+0	7.4870e+0	-1.476e+1
11	-1.013e+0	6.8070e+0	-1.709e+1
12	5.1247e-1	5.5877e+0	-1.964e+1
13	2.0785e+0	3.7456e+0	-2.230e+1
14	4.1725e+0	-2.4275e-2	-2.386e+1
15	1.1826e+1	-1.600e+1	-1.509e+1
16	3.7092e+1	-7.459e+1	3.7069e+1
17	1.1787e+2	-2.811e+2	2.6044e+2
18	3.5586e+2	-9.573e+2	1.0984e+3
19	9.7251e+2	-2.919e+3	3.8492e+3
20	2.1027e+3	-7.009e+3	1.0380e+4
21	2.1290e+3	-7.587e+3	1.2057e+4

TABLE 5: EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 0.25$, FOR x_e

SNR (dB)	Range Lower Limit x_L	Range Upper Limit x_U	A	B	C
0	.01	5.6	-1.5422e-3	4.5899e-2	-5.8885e-1
1	.01	5.77	-1.1524e-3	3.5342e-2	-4.6724e-1
2	.015	5.965	-7.3691e-4	2.3410e-2	-3.2073e-1
3	.02	6.185	-4.6829e-4	1.5451e-2	-2.1993e-1
4	.025	6.435	-2.9259e-4	1.0057e-2	-1.4916e-1
5	.04	6.715	-1.5489e-4	5.5750e-3	-8.6645e-2
6	.065	7.025	-7.4596e-5	2.8221e-3	-4.6143e-2
7	.11	7.375	-3.0750e-5	1.2297e-3	-2.1284e-2
8	.22	7.77	-8.6030e-6	3.6777e-4	-6.8269e-3
9	.45	8.21	-1.3960e-6	6.4832e-5	-1.3188e-3
10	.815	8.705	6.4795e-8	-2.7352e-6	4.0778e-5
11	1.285	9.255	2.3171e-7	-1.2180e-5	2.7321e-4
12	1.84	9.88	2.4897e-7	-1.4384e-5	3.5862e-4
13	2.48	10.575	2.2852e-7	-1.4434e-5	3.9556e-4
14	3.205	11.360	1.7959e-7	-1.2307e-5	3.6627e-4
15	4.03	12.240	1.0541e-7	-7.5467e-6	2.3071e-4
16	4.955	13.225	7.7617e-9	5.1223e-7	-7.3435e-5
17	5.99	14.335	-1.0992e-7	1.2479e-5	-6.2665e-4
18	7.14	15.58	-2.3077e-7	2.7493e-5	-1.4599e-3
19	8.42	16.975	-3.3934e-7	4.4222e-5	-2.5740e-3
20	9.84	18.545	-4.1982e-7	6.0504e-5	-3.8995e-3
21	11.425	20.310	-4.6637e-7	7.4716e-5	-5.3582e-3

TABLE 5 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 0.25$, FOR x_e

SNR (dB)	D	E	F	G	H
0	4.2620e+0	-1.913e+1	5.5130e+1	-1.022e+2	1.1902e+2
1	3.4851e+0	-1.612e+1	4.7874e+1	-9.146e+1	1.0967e+2
2	2.4803e+0	-1.190e+1	3.6714e+1	-7.293e+1	9.1111e+1
3	1.7678e+0	-8.822e+0	2.8317e+1	-5.859e+1	7.6320e+1
4	1.2496e+0	-6.503e+0	2.1771e+1	-4.701e+1	6.3951e+1
5	7.6135e-1	-4.160e+0	1.4649e+1	-3.334e+1	4.7930e+1
6	4.2708e-1	-2.462e+0	9.1652e+0	-2.2111e+1	3.3833e+1
7	2.0893e-1	-1.281e+0	5.0835e+0	-1.313e+1	2.1641e+1
8	7.2173e-2	-4.7894e-1	2.0726e+0	-5.884e+0	1.0776e+1
9	1.5444e-2	-1.1498e-1	5.6590e-1	-1.852e+0	3.9729e+0
10	-1.4745e-4	-2.8661e-3	4.2063e-2	-2.5654e-1	8.7002e-1
11	-3.3979e-3	2.5470e-2	-1.1715e-1	3.2201e-1	-4.7101e-1
12	-5.0431e-3	4.3919e-2	-2.4549e-1	8.8803e-1	-2.052e+0
13	-6.1599e-3	6.0012e-2	-3.8036e-1	1.5864e+0	-4.316e+0
14	-6.1978e-3	6.5611e-2	-4.5101e-1	2.0280e+0	-5.873e+0
15	-3.8945e-3	3.8882e-2	-2.2216e-1	5.5003e-1	1.0567e+0
16	3.0544e-3	-6.7722e-2	9.2140e-1	-8.070e+0	4.5690e+1
17	1.8324e-2	-3.4534e-1	4.3796e+0	-3.783e+1	2.1960e+2
18	4.5471e-2	-9.1937e-1	1.2599e+1	-1.185e+2	7.5399e+2
19	8.8081e-2	-1.962e+0	2.9692e+1	-3.093e+2	2.1882e+3
20	1.4796e-1	-3.659e+0	6.1622e+1	-7.153e+2	5.6515e+3
21	2.2647e-1	-6.246e+0	1.1743e+2	-1.524e+3	1.3479e+4

TABLE 5 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 0.25$, FOR x_e

SNR (dB)	I	J	K
0	-8.324e+1	3.3149e+1	-7.482e+0
1	-7.892e+1	3.2466e+1	-7.792e+0
2	-6.856e+1	2.9934e+1	-7.998e+0
3	-6.005e+1	2.7851e+1	-8.332e+0
4	-5.266e+1	2.6023e+1	-8.80e+0
5	-4.204e+1	2.2912e+1	-9.243e+0
6	-3.193e+1	1.9693e+1	-9.808e+0
7	-2.239e+1	1.6344e+1	-1.052e+1
8	-1.288e+1	1.2518e+1	-1.132e+1
9	-6.047e+0	9.3673e+0	-1.243e+1
10	-2.370e+0	7.5056e+0	-1.409e+1
11	-4.4474e-1	6.5170e+0	-1.637e+1
12	2.2908e+0	4.4562e+0	-1.873e+1
13	6.8048e+0	6.2160e-2	-2.059e+1
14	9.8958e+0	-2.2[illegible]e+0	-2.487e+1
15	-1.177e+1	3.8159e+1	-6.326e+1
16	-1.622e+2	3.3524e+2	-3.290e+2
17	-8.207e+2	1.7913e+3	-1.763e+3
18	-3.110e+3	7.5127e+3	-8.109e+3
19	-1.006e+4	2.7171e+4	-3.276e+4
20	-2.908e+4	8.7994e+4	-1.190e+5
21	-7.775e+4	2.6410e+5	-4.013e+5

TABLE 6: EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 3/8$, FOR x_e

SNR (dB)	Range Lower Limit x_L	Range Upper Limit x_U	A	B	C
0	.015	5.735	-1.0551e-3	3.2194e-2	-4.2355e-1
1	.015	5.915	-7.8236e-4	2.4621e-2	-3.3409e-1
2	.02	6.120	-5.0488e-4	1.6466e-2	-2.3164e-1
3	.025	6.350	-3.2244e-4	1.0926e-2	-1.5975e-1
4	.035	6.610	-1.8673e-4	6.6024e-3	-1.0077e-1
5	.05	6.900	-1.0127e-4	3.7489e-3	-5.9940e-2
6	.08	7.230	-4.6655e-5	1.8189e-3	-3.0660e-2
7	.135	7.595	-1.8049e-5	7.4499e-4	-1.3316e-2
8	.265	8.005	-4.5077e-6	1.9928e-4	-3.8292e-3
9	.515	8.465	-7.2031e-7	3.4402e-5	-7.1993e-4
10	.895	8.980	-6.0718e-8	2.9103e-6	-6.3560e-5
11	1.375	9.560	6.8766e-8	-4.2046e-6	1.0972e-4
12	1.945	10.205	1.0133e-7	-6.5844e-6	1.8570e-4
13	2.605	10.93	1.0839e-7	-7.6542e-6	2.3624e-4
14	3.365	11.745	1.0667e-7	-8.2503e-6	2.8031e-4
15	4.230	12.660	1.0218e-7	-8.6974e-6	3.2652e-4
16	5.210	13.685	9.3045e-8	-8.7357e-6	3.6289e-4
17	6.310	14.835	7.6490e-8	-7.9157e-6	3.6312e-4
18	7.555	16.125	5.1897e-8	-5.8704e-6	2.9418e-4
19	8.945	17.575	2.1102e-8	-2.4552e-6	1.2384e-4
20	10.505	19.20	-1.2266e-8	2.1881e-6	-1.7090e-4
21	12.255	21.03	-4.2599e-8	7.4224e-6	-5.7898e-4

TABLE 6 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 3/8$, FOR x_e

SNR (dB)	D	E	F	G	H
0	3.1452e+0	-1.449e+1	4.2928e+1	-8.197e+1	9.8681e+1
1	2.5587e+0	-1.216e+1	3.7145e+1	-7.312e+1	9.0702e+1
2	1.8401e+0	-9.075e+0	2.8793e+1	-5.894e+1	7.6128e+1
3	1.3193e+0	-6.768e+0	2.2347e+1	-4.764e+1	6.4139e+1
4	8.6934e-1	-4.662e+0	1.6111e+1	-3.60e+1	5.0895e+1
5	5.4208e-1	-3.051e+0	1.1077e+1	-2.605e+1	3.8865e+1
6	2.9271e-1	-1.742e+0	6.7053e+0	-1.677e+1	2.6737e+1
7	1.3509e-1	-8.5685e-1	3.5282e+0	-9.489e+0	1.6392e+1
8	4.1968e-2	-2.8944e-1	1.3072e+0	-3.90e+0	7.5929e+0
9	8.6913e-3	-6.7027e-2	3.4481e-1	-1.197e+0	2.7836e+0
10	8.5778e-4	-8.0720e-3	5.4730e-2	-2.6132e-1	8.4939e-1
11	-1.5920e-3	1.4004e-2	-7.6259e-2	2.5031e-1	-4.3658e-1
12	-2.9748e-3	2.9754e-2	-1.9258e-1	8.1016e-1	-2.164e+0
13	-4.1790e-3	4.6704e-2	-3.4300e-1	1.6704e+0	-5.314e+0
14	-5.4932e-3	6.8557e-2	-5.6782e-1	3.1554e+0	-1.161e+1
15	-7.1065e-3	9.9104e-2	-9.2385e-1	5.8253e+0	-2.454e+1
16	-8.7713e-3	1.3643e-1	-1.426e+0	1.0129e+1	-4.833e+1
17	-9.7131e-3	1.6759e-1	-1.948e+0	1.5427e+1	-8.226e+1
18	-8.5878e-3	1.6147e-1	-2.040e+0	1.7502e+1	-1.006e+2
19	-3.5170e-3	6.0717e-2	-6.2883e-1	3.2831e+0	1.5440e+0
20	7.7315e-3	-2.2503e-1	4.4116e+0	-5.908e+1	5.3408e+2
21	2.6618e-2	-7.9853e-1	1.6329e+1	-2.304e+2	2.2156e+3

TABLE 6 (cont'd): EXPONENTIAL COEFFICIENTS, a = 1.0, b = 3/8, FOR x_e

SNR (dB)	I	J	K
0	-7.202e+1	3.0644e+1	-7.663e+0
1	-6.813e+1	2.9955e+1	-7.973e+0
2	-5.974e+1	2.7835e+1	-8.213e+0
3	-5.259e+1	2.6007e+1	-8.570e+0
4	-4.405e+1	2.3579e+1	-8.960e+0
5	-3.575e+1	2.1040e+1	-9.453e+0
6	-2.663e+1	1.7943e+1	-1.001e+1
7	-1.809e+1	1.4733e+1	-1.070e+1
8	-9.943e+0	1.1226e+1	-1.151e+1
9	-4.812e+0	8.7524e+0	-1.271e+1
10	-2.370e+0	7.5514e+0	-1.451e+1
11	-3.7835e-1	6.3716e+0	-1.668e+1
12	2.9060e+0	3.4739e+0	-1.856e+1
13	1.0025e+1	-4.903e+0	-1.793e+1
14	2.6601e+1	-2.881e+1	-7.431e+0
15	6.5662e+1	-9.507e+1	3.5949e+1
16	1.4789e+2	-2.560e+2	1.6626e+2
17	2.8225e+2	-5.561e+2	4.4864e+2
18	3.6907e+2	-7.714e+2	6.5651e+2
19	-1.287e+2	7.0371e+2	-1.350e+3
20	-3.123e+3	1.0677e+4	-1.627e+4
21	-1.389e+4	5.1234e+4	-8.455e+4

TABLE 7: EXPONENTIAL COEFFICIENTS, $a = 31/32$, $b = 3/8$, FOR x_e

SNR (dB)	Range Lower Limit x_L	Range Upper Limit x_U	A	B	C
0	.01	5.575	-1.5822e-3	4.6842e-2	-5.9773e-1
1	.015	5.75	-1.0277e-3	3.1439e-2	-4.1472e-1
2	.02	5.95	-6.6134e-4	2.0971e-2	-2.8683e-1
3	.025	6.175	-4.2092e-4	1.3871e-2	-1.9723e-1
4	.03	6.43	-2.6300e-4	9.0345e-3	-1.3393e-1
5	.045	6.715	-1.3964e-4	5.0252e-3	-7.8090e-2
6	.075	7.03	-6.3592e-5	2.4088e-3	-3.9441e-2
7	.13	7.39	-2.3968e-5	9.6210e-4	-1.6722e-2
8	.255	7.79	-6.0010e-6	2.5797e-4	-4.8188e-3
9	.5	8.235	-9.5220e-7	4.4183e-5	-8.9789e-4
10	.87	8.74	-9.327e-8	4.3495e-6	-9.1379e-5
11	1.33	9.30	7.3921e-8	-4.5278e-6	1.1780e-4
12	1.885	9.93	1.1898e-7	-7.6182e-6	2.1161e-4
13	2.525	10.635	1.2892e-7	-8.9440e-6	2.7120e-4
14	3.26	11.43	1.2802e-7	-9.7162e-6	3.2398e-4
15	4.10	12.315	1.2580e-7	-1.0500e-5	3.8669e-4
16	5.055	13.315	1.1921e-7	-1.0978e-5	4.4761e-4
17	6.125	14.435	1.0417e-7	-1.0596e-5	4.7828e-4
18	7.335	15.69	7.9951e-8	-8.9591e-6	4.4593e-4
19	8.69	17.1	4.7268e-8	-5.7463e-6	3.0943e-4
20	10.205	18.685	9.6952e-9	-9.9316e-7	3.6398e-5
21	11.905	20.46	-2.6865e-8	4.7589e-6	-3.7602e-4

TABLE 7 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 31/32$, $b = 3/8$, FOR x_e

SNR (dB)	D	E	F	G	H
0	4.3026e+0	-1.920e+1	5.5036e+1	-1.015e+2	1.1772e+2
1	3.0878e+0	-1.427e+1	4.2366e+1	-8.109e+1	9.7818e+1
2	2.2153e+0	-1.062e+1	3.2776e+1	-6.524e+1	8.1971e+1
3	1.5841e+0	-7.903e+0	2.5382e+1	-5.264e+1	6.8951e+1
4	1.1217e+0	-5.838e+0	1.9563e+1	-4.235e+1	5.7924e+1
5	6.8618e-1	-3.751e+0	1.3220e+1	-3.016e+1	4.3596e+1
6	3.6570e-1	-2.113e+0	7.8944e+0	-1.916e+1	2.9609e+1
7	1.6495e-1	-1.017e+0	4.0704e+0	-1.064e+1	1.7857e+1
8	5.1331e-2	-3.4399e-1	1.5092e+0	-4.373e+0	8.2716e+0
9	1.0521e-2	-7.8728e-2	3.9300e-1	-1.325e+0	2.9972e+0
10	1.1689e-3	-1.0313e-2	6.5609e-2	-2.9716e-1	9.2877e-1
11	-1.6974e-3	1.4775e-2	-7.9292e-2	2.5466e-1	-4.2467e-1
12	-3.3371e-3	3.2841e-2	-2.0899e-1	8.6341e-1	-2.259e+0
13	-4.7126e-3	5.1728e-2	-3.7300e-1	1.7824e+0	-5.557e+0
14	-6.2315e-3	7.6337e-2	-6.2056e-1	3.3838e+0	-1.221e+1
15	-8.2585e-3	1.1305e-1	-1.035e+0	6.4082e+0	-2.651e+1
16	-1.0625e-2	1.6243e-1	-1.669e+0	1.1671e+1	-5.484e+1
17	-1.2603e-2	2.1449e-1	-2.462e+0	1.9297e+1	-1.020e+2
18	-1.2972e-2	2.4399e-1	-3.098e+0	2.6888e+1	-1.574e+2
19	-9.7002e-3	1.9560e-1	-2.643e+0	2.4160e+1	-1.468e+2
20	-2.7292e-4	-2.2153e-2	8.8442e-1	-1.648e+1	1.8126e+2
21	1.7454e-2	-5.2713e-1	1.0822e+1	-1.530e+2	1.4694e+3

TABLE 7 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 31/32$, $b = 3/8$, FOR x_e

SNR (dB)	I	J	K
0	-8.237e+1	3.3306e+1	-7.821e+0
1	-7.150e+1	3.0643e+1	-7.946e+0
2	-6.261e+1	2.8453e+1	-8.185e+0
3	-5.505e+1	2.6566e+1	-8.540e+0
4	-4.838e+1	2.4869e+1	-9.022e+0
5	-3.878e+1	2.2004e+1	-9.490e+0
6	-2.857e+1	1.8617e+1	-1.002e+1
7	-1.915e+1	1.5169e+1	-1.070e+1
8	-1.053e+1	1.1564e+1	-1.150e+1
9	-5.058e+0	8.9971e+0	-1.270e+1
10	-2.515e+0	7.7873e+0	-1.450e+1
11	-4.7064e-1	6.6230e+0	-1.670e+1
12	2.9351e+0	3.7027e+0	-1.860e+1
13	1.0234e+1	-4.680e+0	-1.806e+1
14	2.7395e+1	-2.890e+1	-7.690e+0
15	6.9656e+1	-9.909e+1	3.7526e+1
16	1.6534e+2	-2.826e+2	1.8421e+2
17	3.4745e+2	-6.825e+2	5.5860e+2
18	5.9439e+2	-1.30e+3	1.2141e+3
19	5.6268e+2	-1.206e+3	1.0247e+3
20	-1.205e+3	4.5084e+3	-7.352e+3
21	-9.182e+3	3.3716e+4	-5.530e+4

TABLE 8: EXPONENTIAL COEFFICIENTS, $a = 0.948$, $b = 0.393$, FOR x_e

SNR (dB)	Range Lower Limit x_L	Range Upper Limit x_U	A	B	C
0	.01	5.49	-1.8377e-3	5.3571e-2	-6.7308e-1
1	.015	5.665	-1.1871e-3	3.5775e-2	-4.6488e-1
2	.02	5.865	-7.5975e-4	2.3744e-2	-3.2007e-1
3	.025	6.085	-4.8463e-4	1.5736e-2	-2.2044e-1
4	.03	6.34	-3.0099e-4	1.0193e-2	-1.4896e-1
5	.045	6.62	-1.5977e-4	5.6676e-3	-8.6809e-2
6	.075	6.935	-7.2123e-5	2.6945e-3	-4.3514e-2
7	.13	7.29	-2.7062e-5	1.0714e-3	-1.8365e-2
8	.255	7.685	-6.7088e-6	2.8441e-4	-5.2386e-3
9	.495	8.125	-1.0933e-6	4.9927e-5	-9.9742e-4
10	.86	8.625	-1.4511e-7	6.6944e-6	-1.3671e-4
11	1.315	9.18	4.3950e-8	-3.0324e-6	8.5833e-5
12	1.855	9.805	9.5767e-8	-6.3203e-6	1.8027e-4
13	2.485	10.5	1.0725e-7	-7.5820e-6	2.3393e-4
14	3.21	11.285	1.0874e-7	-8.3681e-6	2.8277e-4
15	4.035	12.165	1.1220e-7	-9.4587e-6	3.5192e-4
16	4.975	13.15	1.1575e-7	-1.0742e-5	4.4164e-4
17	6.03	14.255	1.1395e-7	-1.1682e-5	5.3215e-4
18	7.22	15.5	1.0529e-7	-1.1953e-5	6.0411e-4
19	8.55	16.895	8.9424e-8	-1.1243e-5	6.3014e-4
20	10.045	18.46	6.6838e-8	-9.2765e-6	5.7415e-4
21	11.715	20.215	3.9626e-8	-5.9821e-6	4.0160e-4

TABLE 8 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 0.948$, $b = 0.393$, FOR x_e

SNR (dB)	D	E	F	G	H
0	4.7703e+0	-2.096e+1	5.9145e+1	-1.074e+2	1.2262e+2
1	3.4094e+0	-1.552e+1	4.5384e+1	-8.556e+1	1.0166e+2
2	2.4362e+0	-1.151e+1	3.5001e+1	-6.866e+1	8.5004e+[illegible]
3	1.7443e+0	-8.574e+0	2.7124e+1	-5.541e+1	7.1507e+1
4	1.2299e+0	-6.309e+0	2.0839e+1	-4.446e+1	5.9946e+1
5	7.5182e-1	-4.050e+0	1.4068e+1	-3.163e+1	4.5063e+1
6	3.9790e-1	-2.267e+0	8.3527e+0	-1.999e+1	3.0472e+1
7	1.7864e-1	-1.086e+0	4.2861e+0	-1.105e+1	1.8296e+1
8	5.5014e-2	-3.6341e-1	1.5717e+0	-4.492e+0	8.3925e+0
9	1.1476e-2	-8.4234e-2	4.1246e-1	-1.366e+0	3.0493e+0
10	1.6566e-3	-1.3502e-2	7.8663e-2	-3.3111e-1	9.8826e-1
11	-1.3158e-3	1.1992e-2	-6.6464e-2	2.1659e-1	-3.4775e-1
12	-2.9094e-3	2.9215e-2	-1.8910e-1	7.9124e-1	-2.082e+0
13	-4.1304e-3	4.5996e-2	-3.3587e-1	1.6214e+0	-5.087e+0
14	-5.5089e-3	6.8306e-2	-5.6153e-1	3.0925e+0	-1.125e+1
15	-7.5940e-3	1.0504e-1	-9.7154e-1	6.0774e+0	-2.538e+1
16	-1.0580e-2	1.6332e-1	-1.696e+0	1.1991e+1	-5.699e+1
17	-1.4170e-2	2.4407e-1	-2.840e+0	2.2590e+1	-1.213e+2
18	-1.7889e-2	3.4355e-1	-4.469e+0	3.9865e+1	-2.408e+2
19	-2.0724e-2	4.4271e-1	-6.417e+0	6.3885e+1	-4.314e+2
20	-2.0858e-2	4.9230e-1	-7.884e+0	8.6732e+1	-6.469e+2
21	-1.5768e-2	4.0030e-1	-6.853e+0	7.9908e+1	-6.246e+2

TABLE 8 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 0.948$, $b = 0.393$, FOR x_e

SNR (dB)	I	J	K
0	-8.451e+1	3.3738e+1	-7.844e+0
1	-7.323e+1	3.1009e+1	-7.967e+0
2	-6.402e+1	2.8766e+1	-8.205e+0
3	-5.628e+1	2.6850e+1	-8.5595e+0
4	-4.939e+1	2.5114e+1	-9.040e+0
5	-3.954e+1	2.2207e+1	-9.508e+0
6	-2.903e+1	1.8764e+1	-1.004e+1
7	-1.940e+1	1.5275e+1	-1.071e+1
8	-1.060e+1	1.1637e+1	-1.152e+1
9	-5.119e+0	9.1053e+0	-1.272e+1
10	-2.612e+0	7.9441e+0	-1.454e+1
11	-6.0355e-1	6.8294e+0	-1.675e+1
12	2.6179e+0	4.1167e+0	-1.874e+1
13	9.3145e+0	-3.536e+0	-1.858e+1
14	2.5322e+1	-2.619e+1	-9.138e+0
15	6.7212e+1	-9.598e+1	3.5977e+1
16	1.7376e+2	-3.007e+2	2.0102e+2
17	4.2075e+2	-8.438e+2	7.1562e+2
18	9.4237e+2	-2.150e+3	2.1391e+3
19	1.8908e+3	-4.849e+3	5.4842e+3
20	3.1296e+3	-8.856e+3	1.1074e+4
21	3.1156e+3	-8.880e+3	1.0771e+4

TABLE 9: EXPONENTIAL COEFFICIENTS, $a = 0.96043$, $b = 0.39782$, FOR x_e

SNR (dB)	Range Lower Limit x_L	Range Upper Limit x_U	A	B	C
0	.015	5.56	-1.4228e-3	4.2082e-2	-5.3667e-1
1	.015	5.74	-1.0458e-3	3.1934e-2	-4.2043e-1
2	.02	5.94	-6.7273e-4	2.1292e-2	-2.9066e-1
3	.025	6.165	-4.2793e-4	1.4076e-2	-1.9977e-1
4	.035	6.415	-2.4809e-4	8.5119e-3	-1.2606e-1
5	.05	6.7	-1.3346e-4	4.7964e-3	-7.4453e-2
6	.075	7.02	-6.4488e-5	2.4384e-3	-3.9852e-2
7	.135	7.38	-2.3286e-5	9.3387e-4	-1.6217e-2
8	.260	7.78	-5.8691e-6	2.5196e-4	-4.6999e-3
9	.505	8.23	-9.4667e-7	4.3803e-5	-8.8678e-4
10	.870	8.73	-1.3013e-7	6.0822e-6	-1.2580e-4
11	1.335	9.295	3.9229e-8	-2.7342e-6	7.8260e-5
12	1.88	9.925	8.4119e-8	-5.6251e-6	1.6255e-4
13	2.52	10.635	9.4973e-8	-6.7975e-6	2.1234e-4
14	3.255	11.43	9.6305e-8	-7.5038e-6	2.5675e-4
15	4.095	12.32	9.9680e-8	-8.5092e-6	3.2058e-4
16	5.04	13.32	1.0205e-7	-9.5910e-6	3.9933e-4
17	6.11	14.44	1.0037e-7	-1.0422e-5	4.8078e-4
18	7.315	15.695	9.2926e-8	-1.0683e-5	5.4675e-4
19	8.665	17.11	7.8618e-8	-1.0009e-5	5.6808e-4
20	10.18	18.695	5.8443e-8	-8.2119e-6	5.1454e-4
21	11.87	20.475	3.4172e-8	-5.2174e-6	3.5415e-4

TABLE 9 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 0.96043$, $b = 0.39782$, FOR x_e

SNR (dB)	D	E	F	G	H
0	3.8628e+0	-1.725e+1	4.9533e+1	-9.168e+1	1.0700e+2
1	3.1241e+0	-1.440e+1	4.2684e+1	-8.152e+1	9.8112e+1
2	2.2405e+0	-1.072e+1	3.3008e+1	-6.556e+1	8.2179e+1
3	1.6013e+0	-7.973e+0	2.5551e+1	-5.287e+1	6.9095e+1
4	1.0552e+0	-5.490e+0	1.8407e+1	-3.990e+1	5.4754e+1
5	6.5366e-1	-3.571e+0	1.2587e+1	-2.874e+1	4.1638e+1
6	3.6878e-1	-2.127e+0	7.9267e+0	-1.919e+1	2.9595e+1
7	1.5984e-1	-9.8490e-1	3.9402e+0	-1.030e+1	1.7310e+1
8	4.9987e-2	-3.3447e-1	1.4654e+0	-4.244e+0	8.0344e+0
9	1.0341e-2	-7.6957e-2	3.8215e-1	-1.284e+0	2.9082e+0
10	1.5431e-3	-1.2717e-2	7.4842e-2	-3.1804e-1	9.5835e-1
11	-1.2140e-3	1.1202e-2	-6.2904e-2	2.0795e-1	-3.4027e-1
12	-2.6576e-3	2.7033e-2	-1.7723e-1	7.5112e-1	-2.002e+0
13	-3.7962e-3	4.2808e-2	-3.1657e-1	1.5478e+0	-4.919e+0
14	-5.0650e-3	6.3598e-2	-5.2950e-1	2.9536e+0	-1.089e+1
15	-7.0056e-3	9.8143e-2	-9.1939e-1	5.8257e+0	-2.465e+1
16	-9.6877e-3	1.5146e-1	-1.593e+0	1.1404e+1	-5.489e+1
17	-1.2965e-2	2.2616e-1	-2.665e+0	2.1469e+1	-1.168e+2
18	-1.6396e-2	3.1887e-1	-4.200e+0	3.7946e+1	-2.321e+2
19	-1.8919e-2	4.0923e-1	-6.006e+0	6.0547e+1	-4.140e+2
20	-1.8922e-2	4.5206e-1	-7.327e+0	8.1574e+1	-6.157e+2
21	-1.4054e-2	3.6043e-1	-6.229e+0	7.3261e+1	-5.768e+2

TABLE 9 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 0.96043$, $b = 0.39782$ FOR x_e

SNR (dB)	I	J	K
0	-7.579e+1	3.1419e+1	-7.680e+0
1	-7.158e+1	3.0683e+1	-7.988e+0
2	-6.265e+1	2.8480e+1	-8.227e+0
3	-5.506e+1	2.6581e+1	-8.582e+0
4	-4.606e+1	2.4078e+1	-8.971e+0
5	-3.725e+1	2.1451e+1	-9.463e+0
6	-2.851e+1	1.8606e+1	-1.006e+1
7	-1.865e+1	1.4966e+1	-1.071e+1
8	-1.029e+1	1.1467e+1	-1.153e+1
9	-4.960e+0	8.9704e+0	-1.273e+1
10	-2.555e+0	7.8490e+0	-1.456e+1
11	-5.788e-1	6.7328e+0	-1.676e+1
12	2.5493e+0	4.0655e+0	-1.876e+1
13	9.1301e+0	-3.554e+0	-1.856e+1
14	2.4821e+1	-2.605e+1	-9.029e+0
15	6.6140e+1	-9.579e+1	3.6711e+1
16	1.6950e+2	-2.970e+2	2.0113e+2
17	4.1013e+2	-8.329e+2	7.1543e+2
18	9.1994e+2	-2.126e+3	2.1417e+3
19	1.8372e+3	-4.770e+3	5.4618e+3
20	3.0134e+3	-8.625e+3	1.0908e+4
21	2.8923e+3	-8.259e+3	9.9716e+3

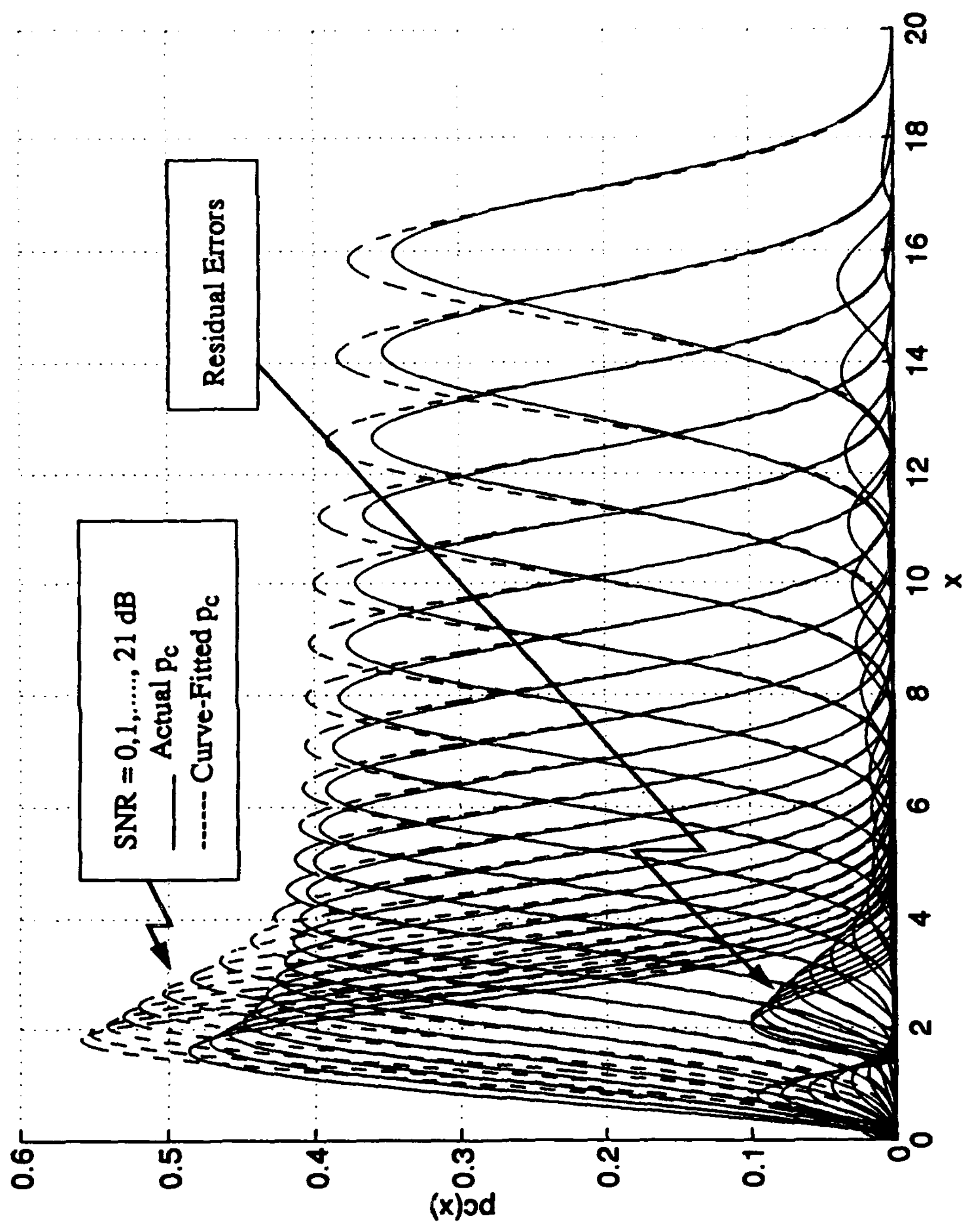

Figure 29: Third Order Curve-Fits for $p_c(x)$, with Residual Errors

IV. PDF FOR THE ENVELOPE DETECTION APPROXIMATION $a|I| + b|Q|$ OF A TARGET TEST CELL

A. CLOSED FORM PDF BY ANALYTICAL METHODS

The envelope detection approximation $\hat{x}_e$ given by (2) is a simplified version of x_e, the approximation considered so far in this thesis. It does not have the *Max* and *Min* operations and can therefore be calculated faster than x_e. Analytically, $\hat{x}_e$ also has the advantage that the probability density function can be derived without the need for error functions. As can be recalled in the pdf development for x_e, the error function terms arose from the expressions for the cumulative distribution functions. These were required to describe the statistical distribution of the *Max* and *Min* operations.

In this situation, the derivation of the density function of $\hat{x}_e$ for a target test cell requires the convolution of two simple linear transformations of the random variables $|I|$ and $|Q|$ by the multiplying coefficients a and b. Applying (44) to equations (21) and (22), the two terms in the convolution are

$$p_{a|I|}(\hat{x}) = \frac{1}{a\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{\hat{x}}{a}-m_I\right)^2} + e^{-\frac{1}{2}\left(\frac{\hat{x}}{a}+m_I\right)^2}\right)u(\hat{x}) \qquad (83)$$

and

$$p_{b|Q|}(\hat{x}) = \frac{1}{b\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{\hat{x}}{b}-m_Q\right)^2} + e^{-\frac{1}{2}\left(\frac{\hat{x}}{b}+m_Q\right)^2}\right)u(\hat{x}) \quad . \qquad (84)$$

These can be convolved since I and Q are assumed to be mutually independent Gaussian random variables. As can be recalled from Section III, the convolution arises from the addition of the two independent components $a|I|$ and $b|Q|$.

The convolution integral is given by

$$p_c(\hat{x}) = \int_{-\infty}^{\infty} p_{a|I|}(y)\, p_{b|Q|}(\hat{x}-y)\, dy$$

or

$$p_c(\hat{x}) = \int_{0}^{\hat{x}} p_{a|I|}(y)\, p_{b|Q|}(\hat{x}-y)\, dy \quad . \tag{85}$$

The integration limits reduce as shown when the unit step functions of (83) and (84) are removed from the integration integral. Expanding (85), the convolution integral becomes

$$p_c(\hat{x}) = \frac{1}{2ab\pi}\int_0^{\hat{x}} \left\{ \left(e^{\left[\frac{2ym_Ib+2\hat{x}m_Qa}{ab}\right]} + e^{\left[\frac{2ym_Ib+2ym_Qa}{ab}\right]} + e^{\left[\frac{2\hat{x}m_Q}{b}\right]} + e^{\left[\frac{2ym_Q}{b}\right]} \right) \times \right.$$

$$\left. e^{-\frac{1}{2a^2b^2}(-2\hat{x}ya^2+y^2b^2+m_I^2a^2b^2+\hat{x}^2a^2+y^2a^2+2ym_Qa^2b+m_Q^2a^2b^2+2\hat{x}m_Qa^2b+2ym_Iab^2)} \right\} dy$$

or

$$p_c(\hat{x}) = \frac{1}{2ab\pi}\left[\int_0^{\hat{x}} e^{\frac{1}{2a^2b^2}[-(a^2+b^2)y^2+(4m_Iab^2+2\hat{x}a^2-2m_Qa^2b-2m_Iab^2)y+(4\hat{x}m_Qa^2b-} \right.$$

$$^{m_I^2a^2b^2-\hat{x}^2a^2-m_Q^2a^2b^2-2\hat{x}m_Qa^2b)]}\, dy +$$

$$\int_0^{\hat{x}} e^{\frac{1}{2a^2b^2}[-(a^2+b^2)y^2+(4m_Iab^2+4m_Qa^2b+2\hat{x}a^2-2m_Qa^2b-2m_Iab^2)y+(-m_I^2a^2b^2-}$$

$$^{\hat{x}^2a^2-m_Q^2a^2b^2-2\hat{x}m_Qa^2b)]}\, dy +$$

$$\int_0^{\hat{x}} e^{\frac{1}{2a^2b^2}[-(a^2+b^2)y^2+(2\hat{x}a^2-2m_Qa^2b-2m_Iab^2)y+(4\hat{x}m_Qa^2b-m_I^2a^2b^2-\hat{x}^2a^2-m_Q^2a^2b^2-2\hat{x}m_Qa^2b)]}\,dy +$$

$$\left.\int_0^{\hat{x}} e^{\frac{1}{2a^2b^2}[-(a^2+b^2)y^2+(4m_Qab^2+2\hat{x}a^2-2m_Qa^2b-2m_Iab^2)y+(-m_I^2a^2b^2-\hat{x}^2a^2-m_Q^2a^2b^2-2\hat{x}m_Qa^2b)]}\,dy\right] \quad . \tag{86}$$

Each of the four integral terms in (86) are now of the form

$$\int_0^{\hat{x}} e^{Ay^2+By+C}\,dy \tag{87}$$

which integrate in closed form to

$$\frac{\sqrt{\pi}}{2\sqrt{-A}}e^{\left[\frac{4CA-B^2}{4A}\right]}\{erf(\frac{B}{2\sqrt{-A}})-erf(\frac{2A\hat{x}+B}{2\sqrt{-A}})\} \quad . \tag{88}$$

Applied to (86), this gives the result

$$p_c(\hat{x}) = \frac{1}{2ab\pi}\left[\frac{\sqrt{\pi}}{2\sqrt{-A}}e^{\left[\frac{4C_1A-B_1^2}{4A}\right]}\{erf\left(\frac{B_1}{2\sqrt{-A}}\right)-erf\left(\frac{2A\hat{x}+B_1}{2\sqrt{-A}}\right)\} +\right.$$

$$\frac{\sqrt{\pi}}{2\sqrt{-A}}e^{\left[\frac{4C_2A-(B_2)^2}{4A}\right]}\{erf\left(\frac{B_2}{2\sqrt{-A}}\right)-erf\left(\frac{2A\hat{x}+B_2}{2\sqrt{-A}}\right)\} +$$

$$\frac{\sqrt{\pi}}{2\sqrt{-A}}e^{\left[\frac{4C_3A-B_3^2}{4A}\right]}\{erf\left(\frac{B_3}{2\sqrt{-A}}\right)-erf\left(\frac{2A\hat{x}+B_3}{2\sqrt{-A}}\right)\} +$$

$$\left.\frac{\sqrt{\pi}}{2\sqrt{-A}}e^{\left[\frac{4C_4A-B_4^2}{4A}\right]}\left\{erf\left(\frac{B_4}{2\sqrt{-A}}\right)-erf\left(\frac{2A\hat{x}+B_4}{2\sqrt{-A}}\right)\right\}\ \right] \tag{89}$$

where

$$A = -(a^2+b^2)\left(\frac{1}{2a^2b^2}\right) \tag{90}$$

$$B_1 = (4m_Iab^2+2\hat{x}a^2-2m_Qa^2b-2m_Iab^2)\left(\frac{1}{2a^2b^2}\right) \tag{91}$$

$$B_2 = (4m_Iab^2+4m_Qa^2b+2\hat{x}a^2-2m_Qa^2b-2m_Iab^2)\left(\frac{1}{2a^2b^2}\right) \tag{92}$$

$$B_3 = (2\hat{x}a^2-2m_Qa^2b-2m_Iab^2)\left(\frac{1}{2a^2b^2}\right) \tag{93}$$

$$B_4 = (4m_Qab^2+2\hat{x}a^2-2m_Qa^2b-2m_Iab^2)\left(\frac{1}{2a^2b^2}\right) \tag{94}$$

$$C_1 = (4\hat{x}m_Qa^2b-m_I^2a^2b^2-\hat{x}^2a^2-m_Q^2a^2b^2-2\hat{x}m_Qa^2b)\left(\frac{1}{2a^2b^2}\right) \tag{95}$$

$$C_2 = (-m_I^2a^2b^2-\hat{x}^2a^2-m_Q^2a^2b^2-2\hat{x}m_Qa^2b)\left(\frac{1}{2a^2b^2}\right) \tag{96}$$

$$C_3 = (4\hat{x}m_Qa^2b-m_I^2a^2b^2-\hat{x}^2a^2-m_Q^2a^2b^2-2\hat{x}m_Qa^2b)\left(\frac{1}{2a^2b^2}\right) \tag{97}$$

$$C_4 = (-m_I^2a^2b^2-\hat{x}^2a^2-m_Q^2a^2b^2-2\hat{x}m_Qa^2b)\left(\frac{1}{2a^2b^2}\right) \tag{98}$$

B. CLOSED FORM PDF BY CURVE-FITTING

Using the same procedure as that used for Section III.B, the density function $p_c(\hat{x})$ is also obtained by numerical convolution for a number of signal-to-noise ratios. The results are curve-fitted using an exponential expansion of the form of (81). The exact coefficient values for each SNR are in Tables 10 to 16 in order to provide a quick and easy reference. This procedure allows the two envelope approximations x_e and $\hat{x}_e$ to be compared in terms of their representing coefficients.

TABLE 10: EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 1.0$ FOR $\hat{x}_e$

SNR (dB)	Range Lower Limit x_L	Range Upper Limit x_U	A	B	C
0	.005	7.48	-5.6415e-5	2.2342e-3	-3.8108e-2
1	.005	7.7	-4.2441e-5	1.7303e-3	-3.0380e-2
2	.005	7.95	-3.1021e-5	1.3057e-3	-2.3666e-2
3	.005	8.23	-2.2070e-5	9.6143e-4	-1.8035e-2
4	.005	8.55	-1.5122e-5	6.8408e-4	-1.3325e-2
5	.005	8.915	-9.9361e-6	4.6842e-4	-9.5072e-3
6	.01	9.325	-5.5082e-6	2.7178e-4	-5.7740e-3
7	.025	9.785	-2.5321e-6	1.3135e-4	-2.9351e-3
8	.09	10.305	-6.6428e-7	3.6566e-5	-8.6903e-4
9	.345	10.895	-2.3983e-8	1.4881e-6	-4.1031e-5
10	.835	11.560	2.2540e-8	-1.3770e-6	3.5360e-5
11	1.425	12.305	2.7403e-8	-1.8370e-6	5.2550e-5
12	2.09	13.145	2.7140e-8	-1.9782e-6	6.1884e-5
13	2.835	14.090	2.3047e-8	-1.8092e-6	6.0928e-5
14	3.655	15.155	1.6102e-8	-1.3247e-6	4.6002e-5
15	4.57	16.355	7.2909e-9	-5.3355e-7	1.3083e-5
16	5.585	17.700	-1.2846e-9	4.0317e-7	-3.4795e-5
17	6.715	19.210	-7.9937e-9	1.2949e-6	-9.0238e-5
18	7.97	20.910	-1.1564e-8	1.9114e-6	-1.3899e-4
19	9.375	22.820	-1.2178e-8	2.1652e-6	-1.7052e-4
20	10.945	24.970	-1.0577e-8	2.0534e-6	-1.7722e-4
21	12.700	27.380	-7.9595e-9	1.6944e-6	-1.6077e-4

TABLE 10 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 1.0$ FOR $\hat{x}_e$

SNR (dB)	D	E	F	G	H
0	3.6625e-1	-2.179e+0	8.3030e+0	-2.027e+1	3.0885e+1
1	3.0056e-1	-1.841e+0	7.2174e+0	-1.813e+1	2.8377e+1
2	2.4169e-1	-1.528e+0	6.1808e+0	-1.601e+1	2.5808e+1
3	1.9059e-1	-1.246e+0	5.2148e+0	-1.396e+1	2.3220e+1
4	1.4619e-1	-9.9219e-1	4.3070e+0	-1.195e+1	2.0573e+1
5	1.0867e-1	-7.6809e-1	3.4707e+0	-1.001e+1	1.7904e+1
6	6.9094e-2	-5.1141e-1	2.4204e+0	-7.315e+0	1.3703e+1
7	3.6968e-2	-2.8825e-1	1.4385e+0	-4.590e+0	9.0966e+0
8	1.1674e-2	-9.7447e-2	5.2293e-1	-1.804e+0	3.8972e+0
9	6.5957e-4	-6.7715e-3	4.5472e-2	-1.9711e-1	5.4568e-1
10	-4.9293e-4	4.0194e-3	-1.9313e-2	5.2758e-2	-6.1962e-2
11	-8.3621e-4	8.1102e-3	-4.9748e-2	1.9631e-1	-4.9075e-1
12	-1.0857e-3	1.1741e-2	-8.1410e-2	3.6778e-1	-1.072e+0
13	-1.1490e-3	1.3298e-2	-9.7665e-2	4.5659e-1	-1.318e+0
14	-8.6775e-4	9.4267e-3	-5.5206e-2	9.8004e-2	7.9852e-1
15	-1.6755e-5	-5.7953e-3	1.3929e-1	-1.657e+0	1.1571e+1
16	1.4906e-3	-3.7844e-2	6.1262e-1	-6.494e+0	4.4898e+1
17	3.5860e-3	-9.0311e-2	1.5092e+0	-1.697e+1	1.2689e+2
18	5.8579e3	-1.5848e-1	2.8755e+0	-3.543e+1	2.9271e+2
19	7.8313e-3	-2.3215e-1	4.6394e+0	-6.327e+1	5.8138e+2
20	8.9499e-3	-2.9274e-1	6.4764e+0	-9.810e+1	1.0044e+3
21	8.9490e-3	-3.2342e-1	7.9258e+0	-1.333e+2	1.5195e+3

TABLE 10 (cont'd): EXPONENTIAL COEFFICIENTS, a = 1.0, b = 1.0 FOR $\hat{x}_e$

SNR (dB)	I	J	K
0	-2.809e+1	1.4681e+1	-5.179e+0
1	-2.648e+1	1.4288e+1	-5.414e+0
2	-2.476e+1	1.3863e+1	-5.713e+0
3	-2.295e+1	1.3408e+1	-6.093e+0
4	-2.102e+1	1.2915e+1	-6.576e+0
5	-1.898e+1	1.2390e+1	-7.189e+0
6	-1.531e+1	1.1092e+1	-7.850e+0
7	-1.092e+1	9.3509e+0	-8.633e+0
8	-5.396e+0	6.8162e+0	-9.493e+0
9	-1.347e+0	4.7240e+0	-1.065e+1
10	-4.6656e-1	4.4785e+0	-1.259e+1
11	3.1129e-1	4.1940e+0	-1.494e+1
12	1.4999e+0	3.4208e+0	-1.768e+1
13	1.7129e+0	4.4670e+0	-2.264e+1
14	-6.206e+0	2.1808e+1	-4.270e+1
15	-4.852e+1	1.1780e+2	-1.421e+2
16	-1.954e+2	4.9081e+2	-5.612e+2
17	-6.054e+2	1.6719e+3	-2.056e+3
18	-1.553e+3	4.7866e+3	-6.542e+3
19	-3.445e+3	1.1896e+4	-1.822e+4
20	-6.651e+3	2.5735e+4	-4.423e+4
21	-1.123e+4	4.8574e+4	-9.350e+4

TABLE 11: EXPONENTIAL COEFFICIENTS, a = 1.0, b = 0.5 FOR $\hat{x}_e$

SNR (dB)	Range Lower Limit x_L	Range Upper Limit x_U	A	B	C
0	.005	6.11	-4.0739e-4	1.3173e-2	-1.8349e-1
1	.005	6.295	-3.0346e-4	1.0113e-2	-1.4517e-1
2	.005	6.5	-2.2156e-4	7.6256e-3	-1.1306e-1
3	.005	6.73	-1.5757e-4	5.6153e-3	-8.6196e-2
4	.005	6.99	-1.0845e-4	4.0133e-3	-6.3966e-2
5	.005	7.285	-7.1773e-5	2.7669e-3	-4.5935e-2
6	.005	7.615	-4.5710e-5	1.8409e-3	-3.1923e-2
7	.015	7.99	-2.1107e-5	8.9384e-4	-1.6306e-2
8	.045	8.41	-7.1578e-6	3.2085e-4	-6.2039e-3
9	.2	8.88	-5.5374e-7	2.7465e-5	-5.9376e-4
10	.605	9.41	-1.1347e-7	6.4902e-6	-1.6294e-4
11	1.14	10.01	-9.2300e-8	5.4751e-6	-1.4238e-4
12	1.775	10.68	-1.1528e-8	7.3671e-7	-2.0688e-5
13	2.505	11.435	6.037e-11	-1.5928e-8	8.5076e-7
14	3.32	12.285	4.216e-10	-4.5317e-8	1.9599e-6
15	4.235	13.235	1.6105e-9	-1.5368e-7	6.3853e-6
16	5.26	14.305	3.3336e-9	-3.3600e-7	1.4918e-5
17	6.405	15.51	5.4627e-9	-5.9855e-7	2.9026e-5
18	7.685	16.86	7.5495e-9	-9.0561e-7	4.8190e-5
19	9.11	18.38	8.5007e-9	-1.1104e-6	6.4304e-5
20	10.7	20.085	6.5009e-9	-8.8275e-7	5.2265e-5
21	12.47	22.0	-8.253e-10	3.4981e-7	-4.3519e-5

TABLE 11 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 0.5$ FOR $\hat{x}_e$

SNR (dB)	D	E	F	G	H
0	1.4406e+0	-7.0065e+0	2.1857e+1	-4.382e+1	5.5080e+1
1	1.1748e+0	-5.889e+0	1.8933e+1	-3.910e+1	5.0587e+1
2	9.4483e-1	-4.891e+0	1.6235e+1	-3.460e+1	4.6122e+1
3	7.4575e-1	-3.996e+0	1.3724e+1	-3.024e+1	4.1612e+1
4	5.7453e-1	-3.195e+0	1.1383e+1	-2.60e+1	3.7020e+1
5	4.2966e-1	-2.488e+0	9.2214e+0	-2.189e+1	3.2366e+1
6	3.1182e-1	-1.885e+0	7.2899e+0	-1.805e+1	2.7784e+1
7	1.6768e-1	-1.068e+0	4.3592e+0	-1.141e+1	1.8652e+1
8	6.7743e-2	-4.5935e-1	2.0043e+0	-5.643e+0	1.0016e+1
9	7.3490e-3	-5.7525e-2	2.9685e-1	-1.020e+0	2.3035e+0
10	2.3593e-3	-2.1763e-2	1.3319e-1	-5.4507e-1	1.4657e+0
11	2.1334e-3	-2.0347e-2	1.2871e-1	-5.4448e-1	1.5139e+0
12	3.3607e-4	-3.4953e-3	2.4246e-2	-1.1255e-1	3.4086e-1
13	-2.0985e-5	2.9302e-4	-2.5621e-3	1.5345e-2	-6.6124e-2
14	-4.5959e-5	6.5555e-4	-6.0484e-3	3.7806e-2	-1.6376e-1
15	-1.5180e-4	2.2859e-3	-2.2859e-2	1.5515e-1	-7.1334e-1
16	-3.8360e-4	6.3234e-3	-6.9900e-2	5.2649e-1	-2.679e+0
17	-8.1963e-4	1.4920e-2	-1.8304e-1	1.5350e+0	-8.705e+0
18	-1.4970e-3	3.0057e-2	-4.0764e-1	3.7849e+0	-2.378e+1
19	-2.1720e-3	4.7343e-2	-6.9534e-1	6.9659e+0	-4.698e+1
20	-1.7601e-3	3.6752e-2	-4.8213e-1	3.7350e+0	-1.254e+1
21	2.7514e-3	-1.0509e-1	2.6089e+0	-4.326e+1	4.7734e+2

TABLE 11 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 0.5$ FOR $\hat{x}_e$

SNR (dB)	I	J	K
0	-4.150e+1	1.7671e+1	-4.652e+0
1	-3.914e+1	1.7202e+1	-4.887e+0
2	-3.669e+1	1.6707e+1	-5.187e+0
3	-3.410e+1	1.6176e+1	-5.568e+0
4	-3.134e+1	1.5602e+1	-6.053e+0
5	-2.842e+1	1.4990e+1	-6.668e+0
6	-2.541e+1	1.4362e+1	-7.449e+0
7	-1.835e+1	1.2085e+1	-8.220e+0
8	-1.106e+1	9.4920e+0	-9.173e+0
9	-3.706e+0	6.4565e+0	-1.029e+1
10	-2.902e+0	6.5895e+0	-1.236e+1
11	-3.069e+0	7.3277e+0	-1.507e+1
12	-1.069e+0	5.9483e+0	-1.750e+1
13	-2.3636e-1	5.5735e+0	-2.103e+1
14	4.4512e-2	5.7907e+0	-2.576e+1
15	1.7067e+0	3.6425e+0	-2.988e+1
16	8.4011e+0	-8.736e+0	-2.633e+1
17	3.1562e+1	-5.952e+1	1.4733e+1
18	9.6423e+1	-2.206e+2	1.8119e+2
19	2.0351e+2	-5.022e+2	4.8969e+2
20	-3.550e+1	4.5997e+2	-1.221e+3
21	-3.373e+3	1.3841e+4	-2.518e+4

TABLE 12: EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 0.25$ FOR $\hat{x}_e$

SNR (dB)	Range Lower Limit x_L	Range Upper Limit x_U	A	B	C
0	.005	5.665	-8.8816e-4	2.6767e-2	-3.4782e-1
1	.005	5.835	-6.6626e-4	2.0687e-2	-2.7696e-1
2	.005	6.025	-4.8777e-4	1.5641e-2	-2.1623e-1
3	.005	6.24	[illegible]7e-4	1.1489e-2	-1.6450e-1
4	.005	6.485	[illegible]	8.1444e-3	-1.2116e-1
5	.005	6.755	-1.5595e-4	5.6048e-3	-8.6829e-2
6	.005	7.06	-9.8414e-5	3.6966e-3	-5.9851e-2
7	.01	7.405	-5.0651e-5	2.0015e-3	-3.4111e-2
8	.025	7.79	-2.1057e-5	8.8247e-4	-1.5975e-2
9	.105	8.23	-5.4285e-6	2.4762e-4	-4.9015e-3
10	.375	8.715	-3.3928e-6	1.6444e-4	-3.4603e-3
11	.795	9.27	-4.7802e-7	2.4862e-5	-5.6183e-4
12	1.38	9.885	4.4594e-8	-2.6391e-6	6.8575e-5
13	2.04	10.58	-1.521e-10	5.1163e-9	4.0578e-8
14	2.775	11.36	-1.0876e-9	7.1638e-8	-1.9784e-6
15	3.6	12.24	-2.5077e-9	1.9182e-7	-6.3354e-6
16	4.525	13.225	-5.6096e-9	4.9085e-7	-1.8818e-5
17	5.55	14.335	-1.2042e-8	1.1958e-6	-5.2465e-5
18	6.7	15.580	-2.441e-8	2.7380e-6	-1.3639e-4
19	7.985	16.975	-4.6017e-8	5.8150e-6	-3.2740e-4
20	9.425	18.545	-7.9046e-8	1.1236e-5	-7.1324e-4
21	11.03	20.310	-1.2136e-7	1.9393e-5	-1.3860e-3

TABLE 12 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 0.25$ FOR $\hat{x}_e$

SNR (dB)	D	E	F	G	H
0	2.5507e+0	-1.160e+1	3.3900e+1	-6.368e+1	7.4832e+1
1	2.0924e+0	-9.806e+0	2.9504e+1	-5.705e+1	6.8917e+1
2	1.6869e+0	-8.161e+0	2.5344e+1	-5.054e+1	6.2891e+1
3	1.3288e+0	-6.655e+0	2.1388e+1	-4.411e+1	5.6686e+1
4	1.0168e+0	-5.289e+0	1.7647e+1	-3.776e+1	5.0307e+1
5	7.5875e-1	-4.109e+0	1.4270e+1	-3.177e+1	4.4015e+1
6	5.4657e-1	-3.094e+0	1.1226e+1	-2.612e+1	3.7817e+1
7	3.2817e-1	-1.959e+0	7.5072e+0	-1.849e+1	2.8422e+1
8	1.6358e-1	-1.042e+0	4.2776e+0	-1.134e+1	1.8905e+1
9	5.5182e-2	-3.8912e-1	1.7836e+0	-5.336e+0	1.0174e+1
10	4.1443e-2	-3.1117e-1	1.5205e+0	-4.858e+0	9.9220e+0
11	7.2312e-3	-5.8378e-2	3.0678e-1	-1.054e+0	2.3062e+0
12	-1.0281e-3	9.8245e-3	-6.2324e-2	2.6471e-1	-7.4149e-1
13	-4.7141e-6	1.0104e-4	-1.0702e-3	6.0659e-3	-1.8244e-2
14	2.9113e-5	-2.3517e-4	8.9522e-4	-1.5983e-4	-9.0667e-3
15	1.1757e-4	-1.3360e-3	9.5165e-3	-4.2298e-2	1.1636e-1
16	4.1422e-4	-5.7651e-3	5.2732e-2	-3.2019e-1	1.2791e+0
17	1.3361e-3	-2.1818e-2	2.3822e-1	-1.759e+0	8.6865e+0
18	3.9680e-3	-7.4579e-2	9.4519e-1	-8.175e+0	4.7653e+1
19	1.0808e-2	-2.3151e-1	3.3602e+0	-3.345e+1	2.2555e+2
20	2.6613e-2	-6.4615e-1	1.0663e+1	-1.211e+2	9.3400e+2
21	5.8329e-2	-1.600e+0	2.9900e+1	-3.852e+2	3.3789e+3

TABLE 12 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 0.25$ FOR $\hat{x}_e$

SNR (dB)	I	J	K
0	-5.1980e+1	1.9376e+1	-4.037e+0
1	-4.9116e+1	1.8898e+1	-4.274e+0
2	-4.6088e+1	1.8394e+1	-4.575e+0
3	-4.2853e+1	1.7858e+1	-4.960e+0
4	-3.9406e+1	1.7296e+1	-5.448e+0
5	-3.5882e+1	1.6743e+1	-6.069e+0
6	-3.2295e+1	1.6216e+1	-6.859e+0
7	-2.6005e+1	1.4709e+1	-7.734e+0
8	-1.9085e+1	1.2864e+1	-8.804e+0
9	-1.2114e+1	1.0796e+1	-1.012e+1
10	-1.2688e+1	1.2192e+1	-1.238e+1
11	-3.4700e+0	6.6301e+0	-1.305e+1
12	8.6893e-1	3.7169e+0	-1.473e+1
13	-4.0848e-1	5.5502e+0	-1.873e+1
14	-4.1072e-1	6.1537e+0	-2.297e+1
15	-6.2911e-1	7.0375e+0	-2.830e+1
16	-3.6775e+0	1.2364e+1	-3.780e+1
17	-2.7938e+1	5.9044e+1	-8.393e+1
18	-1.7971e+2	4.0290e+2	-4.359e+2
19	-9.8619e+2	2.5331e+3	-2.937e+3
20	-4.6847e+3	1.3805e+4	-1.820e+4
21	-1.9309e+4	6.4921e+4	-9.759e+4

TABLE 13: EXPONENTIAL COEFFICIENTS, a = 1.0, b = 3/8 FOR $\hat{x}_e$

SNR (dB)	Range Lower Limit x_L	Range Upper Limit x_U	A	B	C
0	.005	5.865	-5.9627e-4	1.8544e-2	-2.4855e-1
1	.005	6.04	-4.4671e-4	1.4312e-2	-1.9764e-1
2	.005	6.24	-3.2481e-4	1.0754e-2	-1.5346e-1
3	.005	6.46	-2.3131e-4	7.9286e-3	-1.1713e-1
4	.005	6.71	-1.5888e-4	5.6558e-3	-8.6757e-2
5	.005	6.99	-1.0527e-4	3.9019e-3	-6.2319e-2
6	.005	7.31	-6.6350e-5	2.5708e-3	-4.2912e-2
7	.01	7.665	-3.4146e-5	1.3895e-3	-2.4367e-2
8	.035	8.065	-1.1382e-5	4.9132e-4	-9.1576e-3
9	.155	8.515	-1.1236e-6	5.4691e-5	-1.1618e-3
10	.505	9.02	-6.5642e-7	3.4302e-5	-7.8293e-4
11	1	9.59	-3.0786e-7	1.7138e-5	-4.1691e-4
12	1.61	10.23	4.6154e-9	-3.0650e-7	8.9859e-6
13	2.315	10.95	1.1272e-9	-7.7951e-8	2.3756e-6
14	3.105	11.76	-2.835e-11	1.9028e-9	-6.2129e-8
15	3.99	12.67	-4.250e-11	2.9915e-9	-9.9091e-8
16	4.985	13.69	-3.001e-11	1.7139e-9	-3.9367e-8
17	6.095	14.84	3.854e-11	-5.8870e-9	3.4002e-7
18	7.345	16.125	2.262e-10	-2.9381e-8	1.6541e-6
19	8.74	17.575	6.477e-10	-8.8982e-8	5.4011e-6
20	10.31	19.2	1.4930e-9	-2.2408e-7	1.4962e-5
21	12.06	21.03	3.0399e-9	-5.0390e-7	3.7284e-5

TABLE 13 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 1.0$, $b = 3/8$ FOR $\hat{x}_e$

SNR (dB)	D	E	F	G	H
0	1.8793e+0	-8.8111e+0	2.6534e+1	-5.143e+1	6.2603e+1
1	1.5396e+0	-7.437e+0	2.3070e+1	-4.604e+1	5.7630e+1
2	1.2352e+0	-6.165e+0	1.9752e+1	-4.069e+1	5.2494e+1
3	9.7589e-1	-5.040e+0	1.6706e+1	-3.557e+1	4.7362e+1
4	7.5047e-1	-4.023e+0	1.3833e+1	-3.053e+1	4.2078e+1
5	5.6116e-1	-3.131e+0	1.1197e+1	-2.569e+1	3.6758e+1
6	4.0378e-1	-2.353e+0	8.7898e+0	-2.105e+1	3.1420e+1
7	2.4101e-1	-1.478e+0	5.8114e+0	-1.468e+1	2.3179e+1
8	9.6516e-2	-6.3277e-1	2.6753e+0	-7.319e+0	1.2661e+1
9	1.4139e-2	-1.0877e-1	5.5038e-1	-1.845e+0	4.0220e+0
10	1.0246e-2	-8.4819e-2	4.6215e-1	-1.670e+0	3.9248e+0
11	5.8185e-3	-5.1414e-2	2.9937e-1	-1.158e+0	2.9180e+0
12	-1.5279e-4	1.6637e-3	-1.2098e-2	5.9686e-2	-1.9986e-1
13	-4.1764e-5	4.6513e-4	-3.4065e-3	1.6807e-2	-5.8636e-2
14	1.5439e-6	-3.2127e-5	4.7462e-4	-4.2149e-3	1.9021e-2
15	2.3665e-6	-4.7854e-5	7.3611e-4	-7.3315e-3	4.1560e-2
16	8.4746e-7	-2.8198e-5	6.8453e-4	-9.3300e-3	6.9559e-2
17	-1.0117e-5	1.6939e-4	-1.5507e-3	5.7732e-2	1.7941e-2
18	-5.2962e-5	1.0626e-3	-1.3872e-2	1.1852e-1	-6.5125e-1
19	-1.9039e-4	4.3082e-3	-6.5284e-2	6.7028e-1	-4.605e+0
20	-5.8477e-4	1.4805e-2	-2.5358e-1	2.9750e+0	-2.361e+1
21	-1.6210e-3	4.5848e-2	-8.8129e-1	1.1659e+1	-1.048e+2

TABLE 13 (cont'd): EXPONENTIAL COEFFICIENTS, a = 1.0, b = 3/8 FOR $\hat{x}_e$

SNR (dB)	I	J	K
0	-4.558e+1	1.8432e+1	-4.401e+0
1	-4.305e+1	1.7960e+1	-4.637e+0
2	-4.033e+1	1.7446e+1	-4.938e+0
3	-3.749e+1	1.6903e+1	-5.320e+0
4	-3.443e+1	1.6317e+1	-5.806e+0
5	-3.123e+1	1.5703e+1	-6.422e+0
6	-2.788e+1	1.5068e+1	-7.205e+0
7	-2.197e+1	1.3440e+1	-8.069e+0
8	-1.355e+1	1.0656e+1	-9.016e+0
9	-5.857e+0	7.7978e+0	-1.020e+1
10	-6.144e+0	8.7885e+0	-1.246e+1
11	-4.996e+0	8.6299e+0	-1.482e+1
12	1.0981e-4	4.6520e+0	-1.614e+1
13	-2.8703e-1	5.5637e+0	-2.011e+1
14	-4.6302e-1	6.4305e+0	-2.497e+1
15	-5.4542e-1	7.2962e+0	-3.097e+1
16	-6.9732e-1	8.4752e+0	-3.868e+1
17	-6.6461e-1	9.6411e+0	-4.835e+1
18	1.7867e+0	5.6375e+0	-5.539e+1
19	1.9884e+1	-4.111e+1	-1.383e+1
20	1.2096e+2	-3.533e+2	3.9927e+2
21	6.1282e+2	-2.095e+3	3.1247e+3

TABLE 14: EXPONENTIAL COEFFICIENTS, $a = 31/32$, $b = 3/8$ FOR $\hat{x}_e$

SNR (dB)	Range Lower Limit x_L	Range Upper Limit x_U	A	B	C
0	.005	5.715	-7.7098e-4	2.3358e-2	-3.0497e-1
1	.005	5.88	-5.8291e-4	1.8176e-2	-2.4427e-1
2	.005	6.075	-4.2363e-4	1.3651e-2	-1.8959e-1
3	.005	6.29	-3.0135e-4	1.0055e-2	-1.4458e-1
4	.005	6.53	-2.0812e-4	7.2077e-3	-1.0756e-1
5	.005	6.805	-1.3746e-4	4.9591e-3	-7.7080e-2
6	.005	7.115	-8.6894e-5	3.2759e-3	-5.3204e-2
7	.01	7.46	-4.4683e-5	1.7690e-3	-3.0179e-2
8	.035	7.85	-1.4787e-5	6.2088e-4	-1.1256e-2
9	.15	8.29	-1.4578e-6	6.8776e-5	-1.4159e-3
10	.495	8.78	-7.4567e-7	3.8116e-5	-8.5150e-4
11	.975	9.335	-3.8977e-7	2.1162e-5	-5.0221e-4
12	1.57	9.96	1.359e-10	-3.4947e-8	1.7391e-6
13	2.255	10.66	1.5245e-9	-1.0294e-7	3.0647e-6
14	3.025	11.445	-1.539e-10	1.0772e-8	-3.4041e-7
15	3.885	12.330	-1.856e-10	1.4068e-8	-4.8170e-7
16	4.855	13.325	-1.652e-10	1.3146e-8	-4.7352e-7
17	5.935	14.44	-3.901e-11	9.333e-10	6.6338e-8
18	7.15	15.695	3.066e-10	-3.9335e-8	2.1770e-6
19	8.515	17.1	1.0559e-9	-1.4034e-7	8.2474e-6
20	10.04	18.685	2.5021e-9	-3.6314e-7	2.3456e-5
21	11.745	20.46	5.0595e-9	-8.1189e-7	5.8164e-5

TABLE 14 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 31/32$, $b = 3/8$ FOR $\hat{x}_e$

SNR (dB)	D	E	F	G	H
0	2.2460e+0	-1.026e+1	3.0078e+1	-5.678e+1	6.7297e+1
1	1.8517e+0	-8.703e+0	2.6268e+1	-5.100e+1	6.2110e+1
2	1.4851e+0	-7.213e+0	2.2486e+1	-4.507e+1	5.6570e+1
3	1.1725e+0	-5.894e+0	1.9009e+1	-3.939e+1	5.1028e+1
4	9.0511e-1	-4.720e+0	1.5783e+1	-3.388e+1	4.5409e+1
5	6.7543e-1	-3.667e+0	1.2759e+1	-2.848e+1	3.9637e+1
6	4.8705e-1	-2.761e+0	1.0032e+1	-2.336e+1	3.3911e+1
7	2.9035e-1	-1.731e+0	6.6218e+0	-1.626e+1	2.4961e+1
8	1.1537e-1	-7.3540e-1	3.0225e+0	-8.037e+0	1.3510e+1
9	1.6699e-2	-1.2450e-1	6.1069e-1	-1.986e+0	4.2050e+0
10	1.0914e-2	-8.8563e-2	4.7347e-1	-1.680e+0	3.8842e+0
11	6.8405e-3	-5.9020e-2	3.3577e-1	-1.270e+0	3.1320e+0
12	-4.0776e-5	5.5432e-4	-4.7573e-3	2.6969e-2	-1.0335e-1
13	-5.2649e-5	5.7337e-4	-4.1166e-3	2.0040e-2	-6.9377e-2
14	6.7025e-6	-9.4947e-5	9.8840e-4	-6.8804e-3	2.6616e-2
15	1.0190e-5	-1.5284e-4	1.6856e-3	-1.2917e-2	6.0961e-2
16	1.0668e-5	-1.7469e-4	2.1650e-3	-1.9178e-2	1.0926e-1
17	-3.4005e-6	5.7572e-5	-2.7059e-4	-3.7666e-3	5.8466e-2
18	-6.8371e-5	1.3466e-3	-1.7348e-2	1.4813e-1	-8.3369e-1
19	-2.8181e-4	6.1941e-3	-9.1463e-2	9.1942e-1	-6.226e+0
20	-8.8748e-4	2.1772e-2	-3.6181e-1	4.1260e+0	-3.191e+1
21	-2.4490e-3	6.7105e-2	-1.250e+0	1.6043e+1	-1.400e+2

TABLE 14 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 31/32$, $b = 3/8$ FOR $\hat{x}_e$

SNR (dB)	I	J	K
0	-4.774e+1	1.8856e+1	-4.389e+0
1	-4.517e+1	1.8389e+1	-4.626e+0
2.	-4.231e+1	1.7861e+1	-4.927e+0
3	-3.932e+1	1.7303e+1	-5.309e+0
4	-3.615e+1	1.6710e+1	-5.795e+0
5	-3.277e+1	1.6074e+1	-6.4111e+0
6	-2.927e+1	1.5423e+1	-7.194e+0
7	-2.300e+1	1.3733e+1	-8.056e+0
8	-1.407e+1	1.0839e+1	-8.998e+0
9	-5.980e+0	7.9006e+0	-1.018e+1
10	-6.029e+0	8.7579e+0	-1.241e+1
11	-5.260e+0	8.9190e+0	-1.488e+1
12	-2.0839e-1	5.0254e+0	-1.629e+1
13	-2.9033e-1	5.7330e+0	-2.018e+1
.14	-4.9947e-1	6.6552e+0	-2.509e+1
15	-6.0647e-1	7.5690e+0	-3.115e+1
16	-8.0762e-1	8.8203e+0	-3.891e+1
17	-7.5844e-1	9.8783e+0	-4.840e+1
18	2.5662e+0	3.7327e+0	-5.296e+1
19	2.6838e+1	-5.884e+1	7.1317e+0
20	1.5986e+2	-4.606e+2	5.3266e+2
21	7.9581e+2	-2.650e+3	3.8734e+3

TABLE 15: EXPONENTIAL COEFFICIENTS, $a = 0.948$, $b = 0.393$ FOR $\hat{x}_e$

SNR (dB)	Range Lower Limit x_L	Range Upper Limit x_U	A	B	C
0	.005	5.645	-8.744e-4	2.6152e-2	-3.370e-1
1	.005	5.81	-6.586e-4	2.0281e-2	-2.6911e-1
2	.005	6.0	-4.807e-4	1.5289e-2	-2.096e-1
3	.005	6.215	-3.405e-4	1.1219e-2	-1.593e-1
4	.005	6.455	-2.343e-4	8.0149e-3	-1.182e-1
5	.005	6.725	-1.553e-4	5.5331e-3	-8.493e-2
6	.005	7.025	-9.921e-5	3.6904e-3	-5.913e-2
7	.01	7.37	-5.083e-5	1.9863e-3	-3.344e-2
8	.035	7.755	-1.691e-5	7.0022e-4	-1.252e-2
9	.155	8.19	-1.514e-6	7.0211e-5	-1.421e-3
10	.5	8.675	-6.080e-7	3.1075e-5	-6.9511e-4
11	.98	9.22	-3.832e-7	2.0648e-5	-4.867e-4
12	1.57	9.84	-1.168e-8	6.6877e-7	-1.669e-5
13	2.245	10.53	1.5193e-9	-1.036e-7	3.115e-6
14	3.005	11.305	-4.70e-10	3.1455e-8	-9.321e-7
15	3.855	12.18	-4.17e-10	2.9655e-8	-9.330e-7
16	4.81	13.165	-1.71e-10	9.7399e-9	-1.88e-7
17	5.885	14.265	3.798e-10	-4.578e-8	2.3450e-6
18	7.085	15.505	1.4311e-9	-1.70e-7	8.9045e-6
19	8.43	16.895	3.2174e-9	-4.135e-7	2.3613e-5
20	9.93	18.46	5.9566e-9	-8.427e-7	5.3146e-5
21	11.615	20.215	9.7702e-9	-1.530e-6	1.0702e-4

TABLE 15 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 0.948$, $b = 0.393$ FOR $\hat{x}_e$

SNR (dB)	D	E	F	G	H
0	2.4495e+0	-1.104e+1	3.1922e+1	-5.942e+1	6.9432e+1
1	2.0139e+0	-9.343e+0	2.7823e+1	-5.329e+1	6.4009e+1
2	1.6198e+0	-7.761e+0	2.3864e+1	-4.716e+1	5.8369e+1
3	1.2752e+0	-6.327e+0	2.0135e+1	-4.116e+1	5.2596e+1
4	9.8193e-1	-5.056e+0	1.6691e+1	-3.536e+1	4.6763e+1
5	7.3481e-1	-3.937e+0	1.3521e+1	-2.977e+1	4.0872e+1
6	5.3395e-1	-2.985e+0	1.0692e+1	-2.454e+1	3.5082e+1
7	3.1748e-1	-1.867e+0	7.0416e+0	-1.704e+1	2.5757e+1
8	1.2640e-1	-7.9357e-1	3.2098e+0	-8.392e+0	1.3855e+1
9	1.6472e-2	-1.2079e-1	5.8335e-1	-1.871e+0	3.9184e+0
10	8.9342e-3	-7.2830e-2	3.9193e-1	-1.403e+0	3.2817e+0
11	6.5912e-3	-5.6606e-2	3.2100e-1	-1.212e+0	2.9915e+0
12	2.3843e-4	-2.1504e-3	1.2706e-2	-4.8814e-2	1.1409e-1
13	-5.3993e-5	5.9242e-4	-4.2952e-3	2.1380e-2	-7.6762e-2
14	1.6489e-5	-1.9865e-4	1.7034e-3	-9.7771e-3	3.1192e-2
15	1.7543e-5	-2.2598e-4	2.1014e-3	-1.3454e-2	5.0992e-2
16	7.8184e-7	2.0181e-5	-3.0473e-4	2.1525e-3	-1.6839e-2
17	-6.7595e-5	1.2191e-3	-1.4467e-2	1.1590e-1	-6.3258e-1
18	-2.7081e-4	5.2928e-3	-6.9543e-2	6.2408e-1	-3.797e+0
19	-7.8867e-4	1.7058e-2	-2.4976e-1	2.5099e+0	-1.712e+1
20	-1.9670e-3	4.7312e-2	-7.7288e-1	8.6887e+0	-6.643e+1
21	-4.4011e-3	1.1784e-1	-2.147e+0	2.6953e+1	-2.304e+2

TABLE 15 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 0.948$, $b = 0.393$ FOR $\hat{x}_e$

SNR (dB)	I	J	K
0	-4.861e+1	1.9037e+1	-4.423e+0
1	-4.595e+1	1.8558e+1	-4.660e+0
2	-4.308e+1	1.8030e+1	-4.960e+0
3	-4.001e+1	1.7459e+1	-5.342e+0
4	-3.676e+1	1.6850e+1	-5.827e+0
5	-3.334e+1	1.6207e+1	-6.444e+0
6	-2.983e+1	1.5555e+1	-7.227e+0
7	-2.337e+1	1.3817e+1	-8.087e+0
8	-1.418e+1	1.0848e+1	-9.024e+0
9	-5.591e+0	7.7044e+0	-1.018e+1
10	-5.255e+0	8.3106e+0	-1.235e+1
11	-5.080e+0	8.9005e+0	-1.497e+1
12	-6.2182e-1	5.5740e+0	-1.664e+1
13	-2.8564e-1	5.8552e+0	-2.038e+1
14	-5.1125e-1	6.8069e+0	-2.537e+1
15	-5.5837e-1	7.6168e+0	-3.143e+1
16	-3.4763e-1	8.0433e+0	-3.858e+1
17	1.8330e+0	4.3987e+0	-4.363e+1
18	1.4609e+1	-2.4846e+1	-2.326e+1
19	7.5616e+1	-1.873e+2	1.5776e+2
20	3.3041e+2	-9.572e+2	1.1792e+3
21	1.2834e+3	-4.199e+3	6.0691e+3

TABLE 16: EXPONENTIAL COEFFICIENTS, $a = 0.96043$ $b = 0.39783$ FOR $\hat{x}_e$

SNR (dB)	Range Lower Limit x_L	Range Upper Limit x_U	A	B	C
0	.005	5.715	-7.7426e-4	2.3443e-2	-3.0587e-1
1	.005	5.885	-5.8026e-4	1.8099e-2	-2.4325e-1
2	.005	6.075	-4.2514e-4	1.3692e-2	-1.9001e-1
3	.005	6.29	-3.0242e-4	1.0085e-2	-1.4490e-1
4	.005	6.535	-2.0739e-4	7.1837e-3	-1.0721e-1
5	.005	6.805	-1.3815e-4	4.9808e-3	-7.7361e-2
6	.005	7.115	-8.7466e-5	3.2954e-3	-5.3479e-2
7	.01	7.46	-4.5137e-5	1.7854e-3	-3.0428e-2
8	.035	7.85	-1.5076e-5	6.3191e-4	-1.1431e-2
9	.155	8.29	-1.3689e-6	6.4184e-5	-1.3129e-3
10	.51	8.785	-5.4233e-7	2.8055e-5	-6.3508e-4
11	.995	9.34	-3.3557e-7	1.8319e-5	-4.3746e-4
12	1.595	9.96	-9.5417e-9	5.5151e-7	-1.3887e-5
13	2.28	10.665	1.3047e-9	-9.0119e-8	2.7462e-6
14	3.045	11.450	-4.066e-10	2.7606e-8	-8.2906e-7
15	3.91	12.335	-3.630e-10	2.6193e-8	-8.3554e-7
16	4.88	13.330	-1.531e-10	8.9322e-9	-1.8131e-7
17	5.965	14.450	3.247e-10	-3.9778e-8	2.0683e-6
18	7.18	15.705	1.2409e-9	-1.4935e-7	7.9300e-6
19	8.54	17.115	2.7948e-9	-3.6393e-7	2.1058e-5
20	1.0065	18.695	5.2091e-9	-7.4668e-7	4.7713e-5
21	11.77	20.475	8.5565e-9	-1.3578e-6	9.6223e-5

TABLE 16 (cont'd): EXPONENTIAL COEFFICIENTS, a = 0.96043 b = 0.39783 FOR $\hat{x}_e$

SNR (dB)	D	E	F	G	H
0	2.2505e+0	-1.027e+1	3.0060e+1	-5.664e+1	6.7007e+1
1	1.8439e+0	-8.664e+0	2.6135e+1	-5.070e+1	6.1684e+1
2	1.4871e+0	-7.214e+0	2.2459e+1	-4.494e+1	5.6311e+1
3	1.1740e+0	-5.895e+0	1.8988e+1	-3.928e+1	5.0801e+1
4	9.0203e-1	-4.702e+0	1.5715e+1	-3.370e+1	4.5126e+1
5	6.7728e-1	-3.672e+0	1.2761e+1	-2.843e+1	3.9496e+1
6	4.8909e-1	-2.770e+0	1.0046e+1	-2.335e+1	3.3813e+1
7	2.9237e-1	-1.741e+0	6.6435e+0	-1.627e+1	2.4894e+1
8	1.1685e-1	-7.4244e-1	3.0390e+0	-8.040e+0	1.3431e+1
9	1.5385e-2	-1.1400e-1	5.5628e-1	-1.803e+0	3.8132e+0
10	8.2601e-3	-6.8137e-2	3.7103e-1	-1.344e+0	3.1805e+0
11	6.0012e-3	-5.2212e-2	2.9995e-1	-1.147e+0	2.8687e+0
12	1.9991e-4	-1.8148e-3	1.0772e-2	-4.1387e-2	9.5427e-2
13	-4.8227e-5	5.3596e-4	-3.9345e-3	1.9834e-2	-7.2238e-2
14	1.4870e-5	-1.8176e-4	1.5822e-3	-9.2194e-3	2.9841e-2
15	1.5937e-5	-2.0834e-4	1.9667e-3	-1.2785e-2	4.9242e-2
16	1.0186e-6	1.3693e-5	-2.3094e-4	1.6302e-3	-1.4053e-2
17	-6.0467e-5	1.1053e-3	-1.3287e-2	1.0779e-1	-5.9578e-1
18	-2.4438e-4	4.8394e-3	-6.4419e-2	5.8562e-1	-3.609e+0
19	-7.1264e-4	1.5617e-2	-2.3165e-1	2.3583e+0	-1.630e+1
20	-1.7893e-3	4.3607e-2	-7.2179e-1	8.2220e+0	-6.369e+1
21	-4.0094e-3	1.0877e-1	-2.008e+0	2.5545e+1	-2.213e+2

TABLE 16 (cont'd): EXPONENTIAL COEFFICIENTS, $a = 0.96043$ $b = 0.39783$ FOR $\hat{x}_e$

SNR (dB)	I	J	K
0	-4.749e+1	1.8824e+1	-4.438e+0
1	-4.485e+1	1.8342e+1	-4.674e+0
2	-4.208e+1	1.7826e+1	-4.975e+0
3	-3.910e+1	1.7267e+1	-5.358e+0
4	-3.591e+1	1.6660e+1	-5.843e+0
5	-3.260e+1	1.6031e+1	-6.459e+0
6	-2.912e+1	1.5373e+1	-7.241e+0
7	-2.286e+1	1.3669e+1	-8.103e+0
8	-1.390e+1	1.0743e+1	-9.041e+0
9	-5.492e+0	7.6275e+0	-1.020e+1
10	-5.150e+0	8.2227e+0	-1.237e+1
11	-4.936e+0	8.7734e+0	-1.498e+1
12	-5.7984e-1	5.4749e+0	-1.664e+1
13	-2.8179e-1	5.7836e+0	-2.039e+1
14	-4.9793e-1	6.7184e+0	-2.538e+1
15	-5.4500e-1	7.5199e+0	-3.144e+1
16	-3.4588e-1	7.9523e+0	-3.860e+1
17	1.7400e+0	4.4387e+0	-4.373e+1
18	1.4062e+1	-2.412e+1	-2.368e+1
19	7.2911e+1	-1.829e+2	1.5536e+2
20	3.2101e+2	-9.423e+2	1.1761e+3
21	1.2489e+3	-4.140e+3	6.0649e+3

V. COMPARISON OF THE APPROXIMATIONS $aMax\{|I|,|Q|\} + bMin\{|I|,|Q|\}$ AND $a|I| + b|Q|$ OF A TARGET TEST CELL

Signals detected by the envelope detection approximations x_e and $\hat{x}_e$ given by (1) and (2) have significantly different probability density functions. This fact is true for both cases of noise and noise plus target. A graphical comparison of x_e and $\hat{x}_e$ with the density function of the true envelope detector $x = \sqrt{I^2 + Q^2}$ is shown in Figure 30 for the case

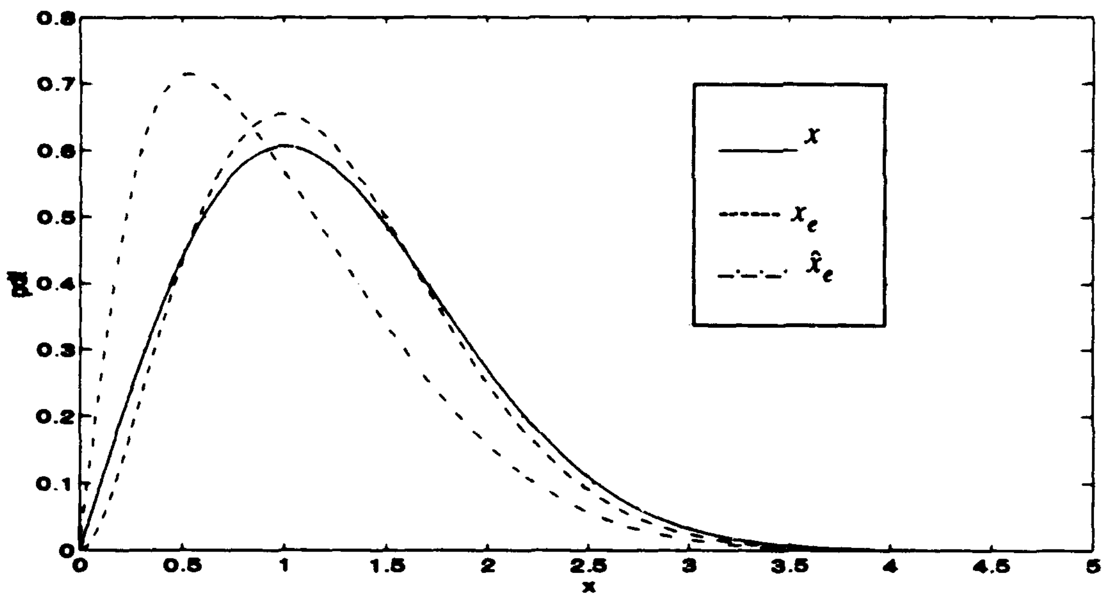

Figure 30: PDF for x, x_e and $\hat{x}_e$, No Target, $n = 1$

in which no target is present. These curves are generated by numerical convolution using the program in Appendix D. The density of x_e is a result of the convolution of (46) and (48).

The density of $\hat{x}_e$ is obtained by convolving equations (83) and (84). Finally, the density function of x is obtained from [Ref. 8]

$$p_{\sqrt{I^2+Q^2}}(x) = xe^{-\frac{x^2}{2}}u(x) \quad . \tag{99}$$

The coded sections for each of these are identified in the program. The multiplying coefficients of both approximations are arbitrarily set at $a = 1.0$ and $b = 0.25$.

The results of the numerical convolutions shown in Figure 30 are verified against experimental results. Data sets generated for I and Q are sorted into histograms, shown in Figure 31. The star symbols " * " represent the true envelope detector x, while the thinner

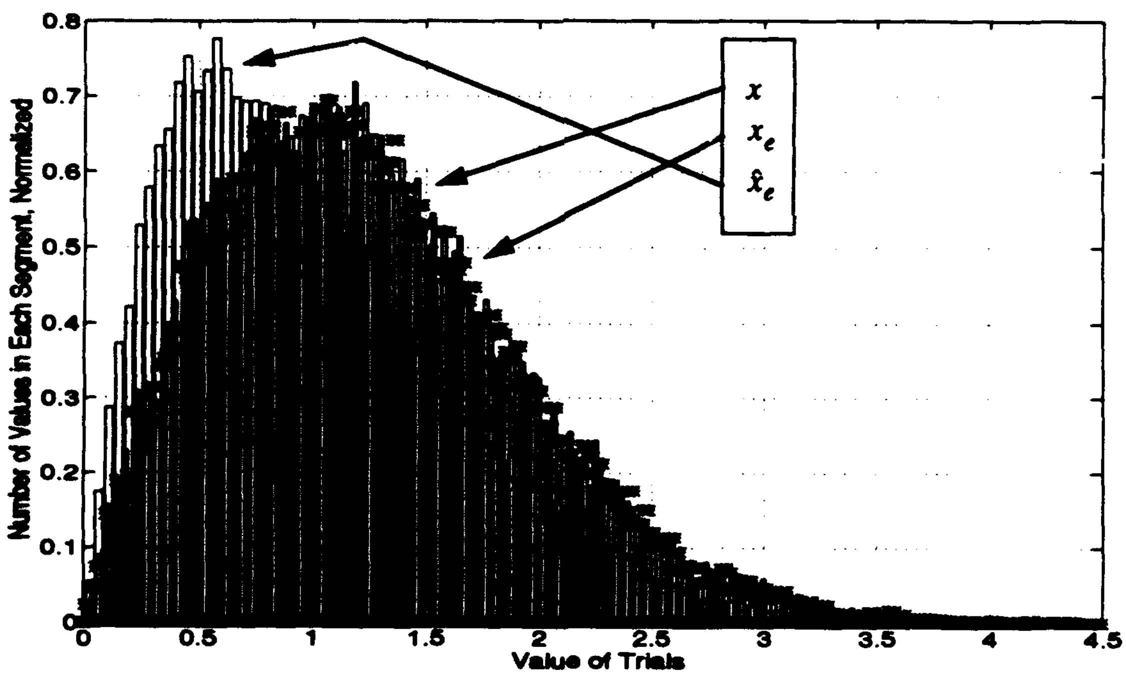

Figure 31: Histograms of x, x_e and $\hat{x}_e$, No Target, $n = 1$

bars represent x_e and the wider bars represent $\hat{x}_e$. The program in Appendix E is used to generate these histograms.

As shown in Figure 30, the envelope detector approximation x_e models the true envelope detector the best. The approximation $\hat{x}_e$ is significantly shifted from the true detector. These curves are used to compute the function $p_{y,n}$ in (7) and (8), for the P_D and the P_{FA} respectively. As explained in Section II.C, $p_{y,n}$ is an n-fold convolution of these curves, for a given n reference cells.

The disparity between $\hat{x}_e$ and x for the function $p_{y,n}$ is enhanced as n is increased. This can be seen in Figures 32 ad 33, where each of x, x_e and $\hat{x}_e$ has been convolved

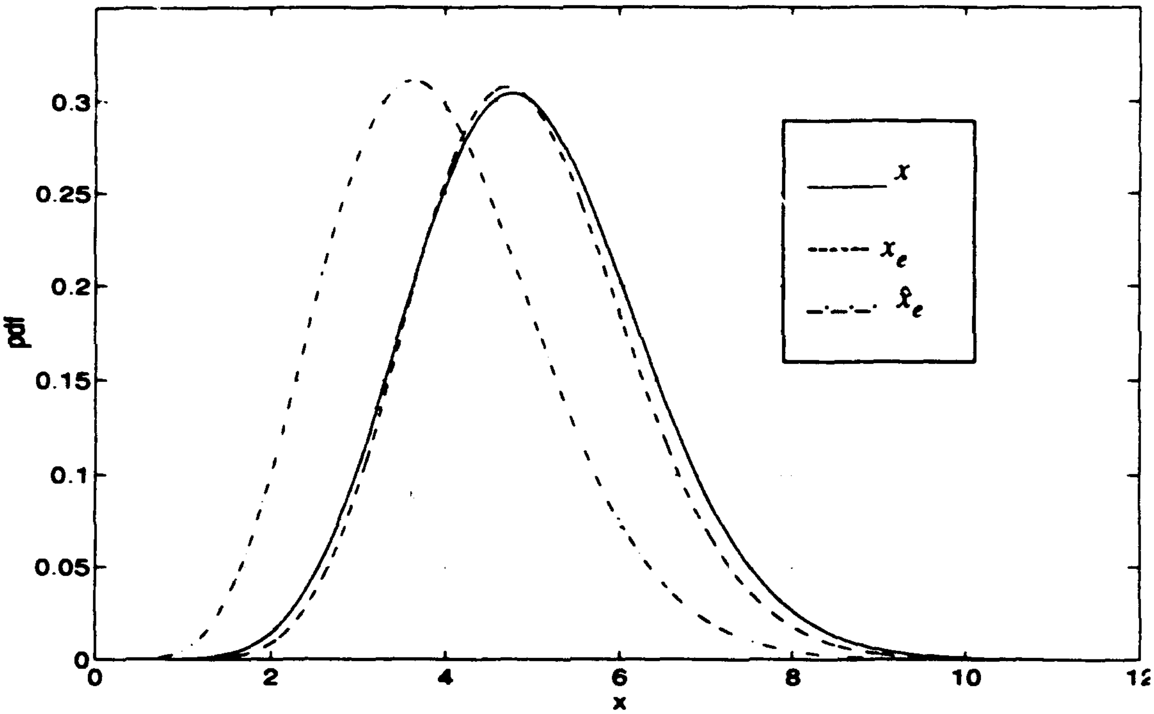

Figure 32: PDF for x, x_e and $\hat{x}_e$, No Target, $n = 4$

for $n = 4$ and $n = 16$. These n-fold convolutions are generated using the program at Appendix F. The approximation x_e generally has the same mean as the true detector x. The mean of the approximation $\hat{x}_e$, however, is increasingly offset from that of the true detector

as n increases. For n = 1, the mean of $\hat{x}_e$ is offset from x by about 0.5 dB in SNR. For n = 4, this offset is about 1.25 dB, whereas for n = 16, the offset is approximately 4 dB.

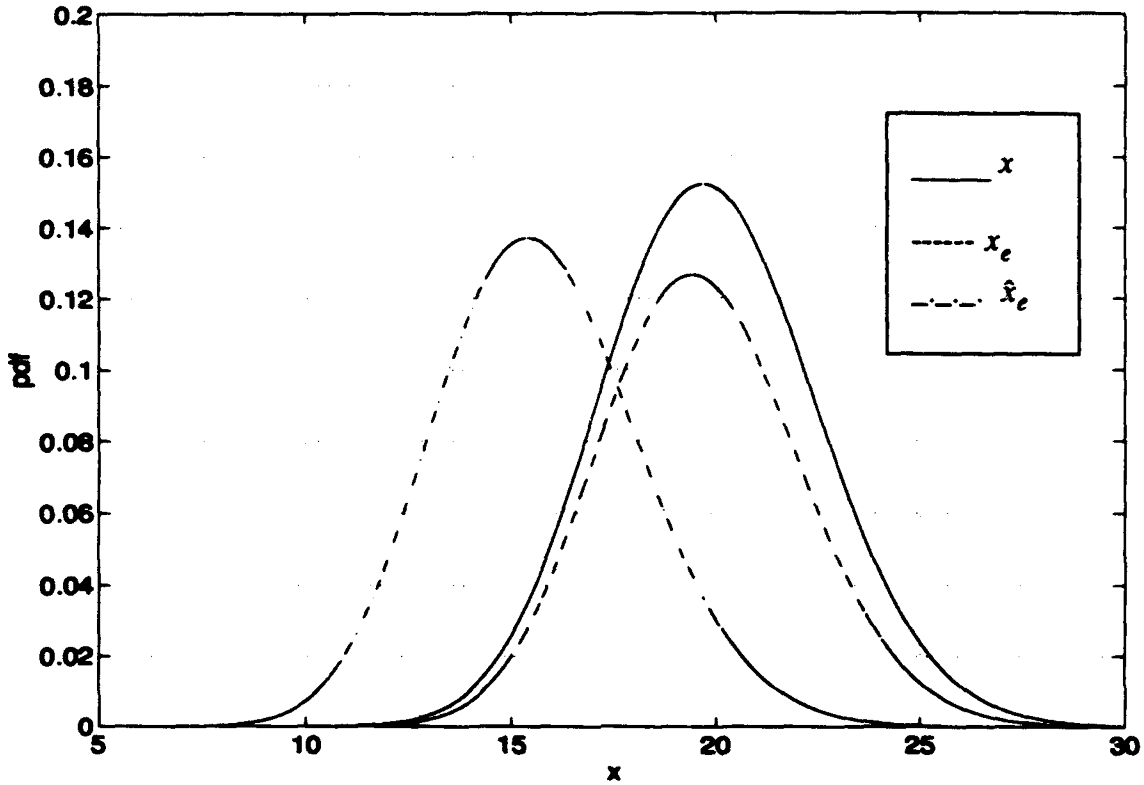

Figure 33: PDF for x, x_e and $\hat{x}_e$, No Target, n = 16

Such differences ultimately create significant differences in the P_D and the P_{FA} GO CFAR performance, as compared to the values obtained for a true envelope detector. As an example, the experimental P_D curves for each of x, x_e and $\hat{x}_e$ are plotted in Figure 34. These are generated using the Monte Carlo simulation program at Appendix G for a = 1.0 and b = 0.25 and n = 4. This results of this simulation program are validated against similar simulations performed in previous research [Ref. 3, Ref. 4]. The thresholds for each detector are chosen so as to yield a $P_{FA} = 10^{-4}$. The threshold for $\hat{x}_e$ is selected from available plots [Ref. 1] as $T_{\hat{x}_e} = 5.537$. The thresholds for x and x_e are determined from

experimental trials using the program at Appendix H. They are set at $T_x = 3.5$ and $T_{x_e} = 4.2$. The P_D curves show that $\hat{x}_e$ provides the most degraded detection sensitivity

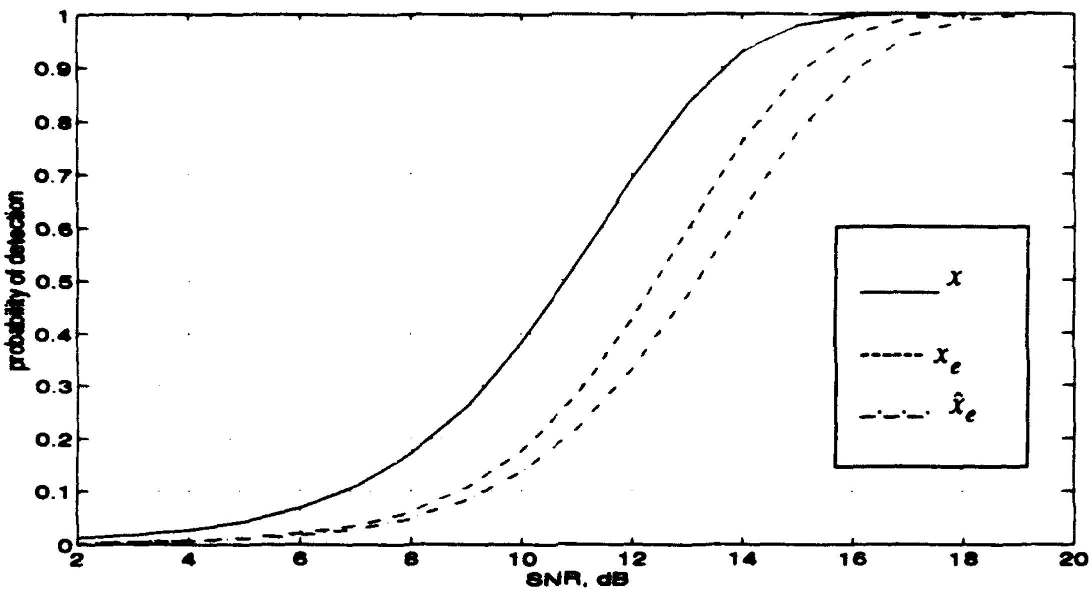

Figure 34: Probability of Detection Curves for x, x_e and $\hat{x}_e$, $a = 1.0$, $b = 0.25$, $n = 4$.

compared to the true envelope detector. The approximation x_e provides a significantly better P_D performance in comparison. As n is increased, this advantage becomes even more pronounced.

VI. CONCLUSION

A. SUMMARY

The main contribution of this thesis is to determine if the probability of detection for a GO CFAR radar processor using the envelope detection approximation $x_e = aMax\{|I|, |Q|\} + bMin\{|I|, |Q|\}$ can be expressed analytically in a closed form. The solution to this problem depends solely on whether the probability density function of a radar range cell p_c can be derived in a closed form for both of the cases where a target is present and where no target is present. It is shown in Section III that this cannot be done without the use of either an error function approximation or an exponential curve-fitting of the density function.

The closed form solution based on an error function approximation is very complex. As stated in Section III.A.6.a, it is comprised of more than 1100 first order exponential terms. For the closed form evaluation of the P_D, each of these terms needs to be integrated in closed form. This approach is not practical. Therefore, the best option is to evaluate the P_D numerically once the error function approximation has been applied to p_c. Although this is still very complex and tedious to program, it has the advantage of being one of the most accurate methods of computing the P_D analytically, as long as the residual error in the error function approximation is kept to a minimum. Another advantage is that this approach is valid for any value of SNR and multiplying coefficients a and b. It can therefore also be used to compute the P_{FA}.

The exponential curve-fitting approach involves the curve-fitting of a number of p_c curves generated numerically for a given range of signal-to-noise ratios. The exponential coefficients resulting from these curve-fits are then summarized in reference tables. This has the advantage of enabling a rapid implementation of p_c within the P_D expression. Then

the P_D can be evaluated numerically, as per the error function approximation approach. The main disadvantage of this method is its lack of versatility. It is dependent upon the multiplying coefficients a and b of the envelope approximation. Furthermore it can only be solved for whole number values of SNR (i.e. 0, 1, 2,.....), up to the SNR required for the $P_D = 1$.

An additional envelope detection approximation $\hat{x}_e = a|I| + b|Q|$ is considered in Section IV. It is offered as an alternative to x_e. Since it is a simplified version of x_e, its density function $p_c(\hat{x})$ is expressed relatively simply in a closed form solution. It is also expressed in the form of exponential curve-fits (as done for x_e), with the required coefficients summarized in reference tables.

Finally, the two envelope detection approximations x_e and $\hat{x}_e$ are compared to each other and to the true envelope detector x. This is done in terms of the density functions for non-target range cells and in terms of the P_D performance. In both instances, it is shown that the approximation x_e delivers a performance which is significantly better than that provided by $\hat{x}_e$. Ultimately, the choice of envelope detection approximation to be used is to be based on P_D performance and the ease of implementation. Although x_e performs better in terms of detection capabilities, $\hat{x}_e$ is simpler to implement since the *Max* and *Min* operations do not have to be performed.

B. FUTURE RESEARCH RECOMMENDATIONS

The possibilities for future research on GO CFAR performance using the envelope detection approximation $x_e = aMax\{|I|, |Q|\} + bMin\{|I|, |Q|\}$ are quite attractive from a mathematical point of view. The ultimate goal of such research would be to provide the radar design and analysis community with a simple method to compute the P_D and the P_{FA} theoretically. This solution would have to be independent of the signal-to-noise ratio in the test cell (or reference cell) and independent of the multiplying coefficients a and b.

Simplified analytical solutions for the P_D and the P_{FA} are invariably contingent upon the ability to integrate expressions of the form of equations (77) and (78) in closed form. This presents a challenging mathematical problem. Solving it would certainly prove to be very useful to anyone using this particular approximation x_e for an envelope detector, whether it is for a GO CFAR processor or for some other use.

Although less glamorous in its possibilities, the ability to approximate the error function in a single segment $-\infty$ to ∞ while maintaining a very small residual error would be very useful. This would greatly reduce the complexity of the closed form probability density function of a test cell as presented in Section III.6. This could be done by appropriately modulating either an inverse tangent function or a hyperbolic tangent function. A closed form solution would then be required for the integral of this function multiplied by an exponential of second order.

Finally, the generation of the P_D curves using the analytical solutions presented in this thesis would be very useful. This could be done for both types of envelope detection approximations x_e and $\hat{x}_e$, for all the cases listed in Table 1. The number of reference cells used should be n = 1, 2, 4, 8, 16 and 32. These represent the majority of the situations encountered in GO CFAR processing.

APPENDIX A - MATLAB PROGRAM FOR POLYNOMIAL CURVE-FITTING OF ERROR FUNCTION

```
%polynomial curve-fitting of error function


x=.01:.01:.5;
y=erf(x);
p1=polyfit(x,y,2);
f=polyval(p1,x);
e=abs(f-y);
plot(x,y,x,f,':'),grid
xlabel('x')
ylabel('erf(x)')

%subplot(212)
%plot(x,e),grid
```

APPENDIX B - C PROGRAM FOR NUMERICAL CONVOLUTION FOR p_c OVER A RANGE OF SNR

```
/*probability density functions of various envelope detectors
 for a GO CFAR   */

#include <stdio.h>
#include <math.h>
#define pi (4.0*atan(1.0))

main(){
  FILE *Pcell, *PcellArea;
  int idelay,c,c1,c2,x,dataLength,s;
  float n;
  double pdfCellTgt[16000],pdfSample,pMax[16000],
       pMin[16000],phi,sum,
       u,snr,ss,a,b,m1,m2,e1,e2,e3,e4,ef1,ef2,ef3,ef4;

  dataLength=16000;
  pdfSample=1.0/200.0;
  a=1.0;
  b=0.25;

  if ((Pcell=fopen("Pyn2.dat","w"))==NULL){
    printf("????");
    exit(1);  }

  for (s=0;s<=29;s++){

    snr=pow(10.0,((float)s/10.0));
    ss=sqrt(2*snr);

    for (c=0;c<dataLength;c++) pdfCellTgt[c]=0.0;

    for (phi=0;phi<=9*pi/40;phi+=pi/40){
      m1=ss*cos(phi);
      m2=ss*sin(phi);

      for (c=0;c<dataLength;c++){
        u=(double)c*pdfSample;

        /*envelope detection approximation aMax{|I|,|Q|}+bMin{|I|,|Q|} */
```

```
        e1=exp(-pow((u/a-m1),2.0)/2.0)+exp(-pow((u/a+m1),2.0)/2.0);
        e2=exp(-pow((u/a-m2),2.0)/2.0)+exp(-pow((u/a+m2),2.0)/2.0);
        e3=exp(-pow((u/b-m1),2.0)/2.0)+exp(-pow((u/b+m1),2.0)/2.0);
        e4=exp(-pow((u/b-m2),2.0)/2.0)+exp(-pow((u/b+m2),2.0)/2.0);
        ef1=erf((u/a-m2)/sqrt(2.0))+erf((u/a+m2)/sqrt(2.0));
        ef2=erf((u/a-m1)/sqrt(2.0))+erf((u/a+m1)/sqrt(2.0));
        ef3=2.0-erf((u/b-m2)/sqrt(2.0))-erf((u/b+m2)/sqrt(2.0));
        ef4=2.0-erf((u/b-m1)/sqrt(2.0))-erf((u/b+m1)/sqrt(2.0));
        pMax[c]=(e1*ef1 + e2*ef2)/(2.0*a*sqrt(2.0*pi));
        pMin[c]=(e3*ef3 + e4*ef4)/(2.0*b*sqrt(2.0*pi));

        /*envelope detection approximation a|I| + b|Q| */

/*       e1=exp(-pow((u/a-m1),2.0)/2.0)+exp(-pow((u/a+m1),2.0)/2.0);
        e4=exp(-pow((u/b-m2),2.0)/2.0)+exp(-pow((u/b+m2),2.0)/2.0);
        pMin[c]=e4/(b*sqrt(2.0*pi));
        pMax[c]=e1/(a*sqrt(2.0*pi));         */

        /*envelope detector sqrt(|I|^2 + |Q|^2) */

/*       pdfCellTgt[c]=u*exp(-pow(u,2.0)/2);  */
      }

      idelay=0;
      for (c=0;c<dataLength;++c){
        idelay+=1;
        sum=0.0;
        for (c1=1;c1<=idelay;++c1)
          sum+=(pMax[c1]*pMin[idelay-c1+1]);
        pdfCellTgt[c]+=(sum*pdfSample);
      }

    }     /****************END OF PHASE LOOP*********************/

    for (x=0;x<dataLength;x++){
      pdfCellTgt[x]=pdfCellTgt[x]/10.0;
      fprintf(Pcell,"%e\n",pdfCellTgt[x]);
    }

  }               /*****END OF SNR LOOP**************************/
  fclose(Pcell);
} /******************************************* ******** END MAIN */
```

APPENDIX C - MATLAB PROGRAM FOR EXPONENTIAL CURVE-FITTING OF p_c FOR A RANGE OF SNR

```
%curve-fitting of target test cell pdf curves over
%a range of SNR values (in dB)

load e1
c=e1;
clg
clear p
hold
order=10;
SNR=21;
sampleSize=1/200;
for q=1:SNR+1,
  r1(q)=1;
  while c(r1(q),q)<.00001,
    r1(q)=r1(q)+1;
  end
  r2(q)=r1(q)+100;
  while c(r2(q),q)>.00001,
    r2(q)=r2(q)+1;
  end
  x=[r1(q)-1:r2(q)-1]'.*sampleSize;
  p(q,:)=polyfit(x,log(c(r1(q):r2(q),q)),order);
  f=exp(polyval(p(q,:),x));
  plot(x,c(r1(q):r2(q),q),x,f,'--',x,abs(f-c(r1(q):r2(q),q)))
end
hold
```

APPENDIX D - C PROGRAM FOR p_c WITH NO TARGET

```
/*probability density functions of various envelope detectors
  for a GO CFAR for no target case    */

#include <stdio.h>
#include <math.h>
#define pi (4.0*atan(1.0))

main(){
  FILE *Pcell, *PcellArea;
  int idelay,c,c1,c2,x,dataLength,s;
  float n;
  double pdfCellTgt[16000],pdfSample,pMax[16000],
       pMin[16000],phi,sum,
       u,snr,ss,a,b,m1,m2,e1,e2,e3,e4,ef1,ef2,ef3,ef4;

  dataLength=16000;
  pdfSample=1.0/200.0;
  a=1.0;
  b=0.25;

  if ((Pcell=fopen("Pyn2.dat","w"))==NULL){
    printf("????");
    exit(1);  }

  ss=0.0;

  for (c=0;c<dataLength;c++) pdfCellTgt[c]=0.0;

  m1=ss;
  m2=ss;

  for (c=0;c<dataLength;c++){
    u=(double)c*pdfSample;

    /*envelope detection approximation aMax{|I|,|Q|}+bMin{|I|,|Q|} */

    e1=exp(-pow((u/a-m1),2.0)/2.0)+exp(-pow((u/a+m1),2.0)/2.0);
    e2=exp(-pow((u/a-m2),2.0)/2.0)+exp(-pow((u/a+m2),2.0)/2.0);
```

```
      e3=exp(-pow((u/b-m1),2.0)/2.0)+exp(-pow((u/b+m1),2.0)/2.0);
      e4=exp(-pow((u/b-m2),2.0)/2.0)+exp(-pow((u/b+m2),2.0)/2.0);
      ef1=erf((u/a-m2)/sqrt(2.0))+erf((u/a+m2)/sqrt(2.0));
      ef2=erf((u/a-m1)/sqrt(2.0))+erf((u/a+m1)/sqrt(2.0));
      ef3=2.0-erf((u/b-m2)/sqrt(2.0))-erf((u/b+m2)/sqrt(2.0));
      ef4=2.0-erf((u/b-m1)/sqrt(2.0))-erf((u/b+m1)/sqrt(2.0));
      pMax[c]=(e1*ef1 + e2*ef2)/(2.0*a*sqrt(2.0*pi));
      pMin[c]=(e3*ef3 + e4*ef4)/(2.0*b*sqrt(2.0*pi));

      /*envelope detection approximation a|I| + b|Q| */

  /*    e1=exp(-pow((u/a-m1),2.0)/2.0)+exp(-pow((u/a+m1),2.0)/2.0);
      e4=exp(-pow((u/b-m2),2.0)/2.0)+exp(-pow((u/b+m2),2.0)/2.0);
      pMin[c]=e4/(b*sqrt(2.0*pi));
      pMax[c]=e1/(a*sqrt(2.0*pi));          */

      /*envelope detector sqrt(|I|^2 + |Q|^2) */

  /*    pdfCellTgt[c]=u*exp(-pow(u,2.0)/2);   */
    }

    idelay=0;
    for (c=0;c<dataLength;++c){
      idelay+=1;
      sum=0.0;
      for (c1=1;c1<=idelay;++c1)
        sum+=(pMax[c1]*pMin[idelay-c1+1]);
      pdfCellTgt[c]+=(sum*pdfSample);
    }

    for (x=0;x<dataLength;x++){
      pdfCellTgt[x]=pdfCellTgt[x]/10.0;
      fprintf(Pcell,"%e\n",pdfCellTgt[x]);
    }

    fclose(Pcell);
  } /******************************************************** END MAIN */
```

APPENDIX E - MATLAB PROGRAM COMPARISON OF ENVELOPE DETECTION APPROXIMATION METHODS BY HISTOGRAM

```
%comparison of various envelope detectors
%for GO CFAR by histogram

clear
clg
b=.25;
a=1;
N=30000;
q=randn(1,N);
i=randn(1,N);
Q=abs(q);
I=abs(i);
z=sqrt(i.^2 + q.^2);
z1=a*max(I,Q)+b*min(I,Q);
z2=a*I + b*Q;
[m,n]=hist(z,200);
[m1,n1]=hist(z1,200);
[m2,n2]=hist(z2,100);
bar(n,m./((1/48)*N),'*')
hold
bar(n1,m1./((1/48)*N))
bar(n2,m2./((1/24)*N))
hold
```

APPENDIX F - C PROGRAM FOR n-FOLD CONVOLUTION FOR $p_{y,n}$

```
/*n-fold convolution of reference cell PDF for a GO-CFAR */

#include <stdio.h>
#include <math.h>
#define pi (4.0*atan(1.0))

main(){
  FILE *Pcell, *PcellArea;
  int idelay,c,c1,c2,x,dataLength;
  float s,n;
  double pdfCellTgt[16000],pdfSample,phaseSample,pMax[16000],
       p2[16000],p1[16000],p3[16000],p4[16000],p5[16000],
       pMin[16000],phi,theta,sumPhase[16000],area,sum,
       u,snr,ss,a,b,m1,m2,e1,e2,e3,e4,ef1,ef2,ef3,ef4;

  dataLength=16000;
  pdfSample=1.0/200.0;
  a=1.0;
  b=.25;

  if ((Pcell=fopen("pyn.dat","w"))==NULL){
    printf("????");
    exit(1);  }

  ss=0.0;
  m1=ss;
  m2=ss;

  for (c=0;c<dataLength;c++){
    u=(double)c*pdfSample;

    /*envelope detection approximation aMax{|I|,|Q|}+bMin{|I|,|Q|} */

/*    e1=exp(-pow((u/a-m1),2.0)/2.0)+exp(-pow((u/a+m1),2.0)/2.0);
    e2=exp(-pow((u/a-m2),2.0)/2.0)+exp(-pow((u/a+m2),2.0)/2.0);
    e3=exp(-pow((u/b-m1),2.0)/2.0)+exp(-pow((u/b+m1),2.0)/2.0);
    e4=exp(-pow((u/b-m2),2.0)/2.0)+exp(-pow((u/b+m2),2.0)/2.0);
    ef1=erf((u/a-m2)/sqrt(2.0))+erf((u/a+m2)/sqrt(2.0));
    ef2=erf((u/a-m1)/sqrt(2.0))+erf((u/a+m1)/sqrt(2.0));
```

```
    ef3=2.0-erf((u/b-m2)/sqrt(2.0))-erf((u/b+m2)/sqrt(2.0));
    ef4=2.0-erf((u/b-m1)/sqrt(2.0))-erf((u/b+m1)/sqrt(2.0));
    pMax[c]=(e1*ef1 + e2*ef2)/(2.0*a*sqrt(2.0*pi));
    pMin[c]=(e3*ef3 + e4*ef4)/(2.0*b*sqrt(2.0*pi));     */

    /*envelope detection approximation a|I| + b|Q| */

 /*    e1=exp(-pow((u/a-m1),2.0)/2.0)+exp(-pow((u/a+m1),2.0)/2.0);
    e4=exp(-pow((u/b-m2),2.0)/2.0)+exp(-pow((u/b+m2),2.0)/2.0);
    pMin[c]=e4/(b*sqrt(2.0*pi));
    pMax[c]=e1/(a*sqrt(2.0*pi));         */

    /*true envelope detector sqrt(|I|^2 + |Q|^2) */

    p1[c]=u*exp(-pow(u,2.0)/2);

  }

 /*do not perform this convolution for true envelope detector  */
 /*  idelay=0;
   for (c=0;c<dataLength;++c){
     p1[c]=0.0;
     idelay+=1;
     sum=0.0;
     for (c1=1;c1<=idelay;++c1)
       sum+=(pMax[c1]*pMin[idelay-c1+1]);
     p1[c]=sum*pdfSample;
   }                    */

 /*convolve up to here for n=1  */

   idelay=0;
   for (c=0;c<dataLength;++c){
     p2[c]=0.0;
     idelay+=1;
     sum=0.0;
     for (c1=1;c1<=idelay;++c1)
       sum+=(p1[c1]*p1[idelay-c1+1]);
     p2[c]=sum*pdfSample;
   }

 /*convolve up to here for n=2  */
```

```
  idelay=0;
  for (c=0;c<dataLength;++c){
    p3[c]=0.0;
    idelay+=1;
    sum=0.0;
    for (c1=1;c1<=idelay;++c1)
      sum+=(p2[c1]*p2[idelay-c1+1]);
    p3[c]=sum*pdfSample;
  }

/*convolve up to here for n=4   */

  idelay=0;
  for (c=0;c<dataLength;++c){
    p4[c]=0.0;
    idelay+=1;
    sum=0.0;
    for (c1=1;c1<=idelay;++c1)
      sum+=(p3[c1]*p3[idelay-c1+1]);
    p4[c]=sum*pdfSample;
/*    fprintf(Pcell,"%e\n",p4[c]);   */
  }

/*convolve up to here for n=8   */

  idelay=0;
  for (c=0;c<dataLength;++c){
    p5[c]=0.0;
    idelay+=1;
    sum=0.0;
    for (c1=1;c1<=idelay;++c1)
      sum+=(p4[c1]*p4[idelay-c1+1]);
    p5[c]=sum*pdfSample;
    fprintf(Pcell,"%e\n",p5[c]);
  }

/*convolve up to here for n=16   */

  fclose(Pcell);

} /************************************************** END MAIN */
```

APPENDIX G - MATLAB PROGRAM FOR MONTE CARLO SIMULATIONS OF P_D

```
%GO CFAR Monte Carlo simulation for probability of detection
clear

T=3.5;
T1=4.2;
T2=5.537;
a=1;
b=.25;
n=4;
S=10000;
PHI=pi/4;
phi=0:PHI/20:PHI;
snr=0:25;
w=2*n;

for B=1:length(phi),
 for C=1:length(snr),
  SNR=10^(snr(C)/10);
  A=sqrt(2*SNR);
  d=0;
  d1=0;
  d2=0;
  X=zeros(1,S);

  I=A*cos(phi(B)) + randn(1,S);
  Q=A*sin(phi(B)) + randn(1,S);
  X=sqrt(I.^2 + Q.^2);
  X1=a*max(abs(I),abs(Q)) + b*min(abs(I),abs(Q));
  X2=a*abs(I) + b*abs(Q);

  i=randn(S,w);
  q=randn(S,w);
  x=sqrt(i.^2 + i.^2);
  x1=a*max(abs(i),abs(q)) + b*min(abs(i),abs(q));
  x2=a*abs(i) + b*abs(q);

  ld=sum(x(:,1:n)');
  lg=sum(x(:,n+1:w)');
  z=max(ld,lg);
```

```
    Vt=T*z/n;

    ld1=sum(x1(:,1:n)');
    lg1=sum(x1(:,n+1:w)');
    z1=max(ld1,lg1);
    Vt1=T1*z1/n;

    ld2=sum(x2(:,1:n)');
    lg2=sum(x2(:,n+1:w)');
    z2=max(ld2,lg2);
    Vt2=T2*z2/n;

    d=length(find((X-Vt)>0));
    Pd(B,C)=d/S;
    d1=length(find((X1-Vt1)>0));
    Pd1(B,C)=d1/S;
    d2=length(find((X2-Vt2)>0));
    Pd2(B,C)=d2/S;
  end
end

PD=mean(Pd);
PD1=mean(Pd1);
PD2=mean(Pd2);

save PD PD PD1 PD2
quit
```

APPENDIX H - MATLAB PROGRAM FOR MONTE CARLO SIMULATIONS OF P_{FA}

```
%GO CFAR Monte Carlo simulation for probability of
%false alarm
clear

T=3.5;
T1=4.2;
a=1;
b=.25;
n=4;
S=100000;
w=2*n;

d=0;
d1=0;
X=zeros(1,S);

I=randn(1,S);
Q=randn(1,S);
X=sqrt(I.^2 + Q.^2);
X1=a*max(abs(I),abs(Q)) + b*min(abs(I),abs(Q));

i=randn(S,w);
q=randn(S,w);
x=sqrt(i.^2 + i.^2);
x1=a*max(abs(i),abs(q)) + b*min(abs(i),abs(q));

ld=sum(x(:,1:n)');
lg=sum(x(:,n+1:w)');
z=max(ld,lg);
Vt=T*z/n;

ld1=sum(x1(:,1:n)');
lg1=sum(x1(:,n+1:w)');
z1=max(ld1,lg1);
Vt1=T1*z1/n;

d=length(find((X-Vt)>0));
Pfa=d/S;
d1=length(find((X1-Vt1)>0));
```

```
Pfa1=d1/S;

save PFA Pfa Pfa1
quit
```

LIST OF REFERENCES

1. Pace, P.E., and Taylor, L.L., "False Alarm Analysis of the Envelope Detection GO-CFAR Processor," *IEEE Transactions on Aerospace and Electronics Systems*, Vol. 30, No. 3, pp. 848-864, July 1994.

2. General Dynamics Technical Memorandum 6-257-640, *Probability of Detection of the Sum of the Magnitudes of In-phase and Quadrature Signals in Gaussian Noise*, by D.J. Wilson, 4 April 1982.

3. General Dynamics Technical Memorandum 6-337-117.36-3, *Inclusion of Target Cell In Cell Averaging CFAR*, by D.G. Loberger, 17 October 1978.

4. Hughes Missile Systems Company Technical Memorandum 41-3-198.64-1, *Probability of Detection and Probability of False Alarm for the Phalanx Block I GO-CFAR Signal Processor*, by G.P. Laulusa, 19 November 1992.

5. Hansen, V.G., and Sawyers, J.H., "Detectability Loss Due to 'Greatest Of' Selection in a Cell-Averaging CFAR," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-16, No. 1, pp. 115-118, January 1980.

6. Filip, A.E., "A Baker's Dozen Magnitude Approximations and Their Detection Statistics," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. AES-12, No. 1, pp. 86-89, January 1976.

7. Nathanson, F.E., and Luke, P.J., "Loss from Approximations to Square-Law Detectors in Quadrature Systems with Postdetection Integration," *IEEE Transactions on Aerospace and Electronic Systems*, pp. 75-77, January 1972.

8. Papoulis, A., *Probability, Random Variables, and Stochastic Processes*, 3rd ed., McGraw-Hill, Inc., 1991.

9. Spiegel, M.R., *Schaum's Outline Series, Theory and Problems of: Mathematical Handbook of Formulas and Tables*, p.53, McGraw-Hill Book Co., 1968.

10. Abramowitz, M., and Stegan, I.A., *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, pp. 295-305, June 1964.

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